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Large Non-Cooperative Games: Foundations and Tools

by

Georgios Stergianopoulos

Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

in Interdisciplinary Mathematics

Supervisors: Professors Peter J. Hammond and Robert S. MacKay

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August 2012

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I would like to thank the examiners Professors Andrés Carvajal and Michael Dempster for providing me with revision guidelines that helped improve my PhD dissertation.

This note includes a list of changes that follow closely the examiners' guidelines. The changes listed below are related with providing better motivation, clarifying, and correcting, and do *not* include typographical corrections and other changes of minor importance.

Chapter 1

Page 5 Section 1.2: added formal definitions of (1) upper hemicontinuity, and (2) lower hemicontinuity of correspondences.

Page 9 Section 1.3.1: corrected the interpretation of a measure μ on the space of strategies of all players

Page 9 Section 1.3.1: added in the definition of a game with a continuum of players the corresponding sigma-algebra

Pages 9-10 Section 1.3.1: added remark clarifying that in a game with a continuum of players the payoff function of each player is assumed to be a continuous function.

Pages 10-11 Section 1.3.1: added a page clarifying the motivation for studying purely-aggregative games using the Schmeidler (1973) framework: our goal is to show that, irrespective of the intent of Schmeidler, his framework is not suitable for the study of purely-aggregative games with infinitely-many players.

Page 11 Section 1.3.1: added a remark acknowledging that the proof of Proposition 1 is immediate and clarifying that its primary purpose is not to establish a ground-breaking result but to facilitate understanding of the related concepts.

Page 11 Section 1.3.1: added a remark acknowledging that Proposition 1 is restricted to purely-aggregative games while our examples of (1) Cournot market game, and (2) welfare maximization in a multi-period economy, are general aggregative games.

Page 13 Section 1.3.2: clarified that in a restricted game with finitely-many players the payoff function of each player is assumed to be a continuous function.

Pages 15-16 Section 1.4: added formal definitions of (1) "economic negligibility" and (2) "strategic insignificance" that reflect the fact that the former concept is defined with respect to a given aggregator function, whereas the latter is not.

Pages 24-25 Section 1.6: added formal definition of a "better-response" that distinguishes this concept from the related preference relation in the next definition.

Pages 25-26 Section 1.6: corrected the definition of a "better-response preference relation" over strategies and clarified that is a family of binary relations indexed by distributions of strategies of all other players.

Page 26 Section 1.6: clarified that the "better-response preference relation" over strategies is an asymmetric relation and is induced by a continuous payoff function.

Pages 26-27 Section 1.6: corrected the definition of a “better-response overtaking preference relation” over strategies to reflect the fact that (1) it is a family of binary relations indexed by distributions of strategies, (2) it involves all sequences of distributions of strategies that converge to μ.

Page 27 Section 1.6: clarified that the “better-response overtaking preference relation” over strategies is an asymmetric relation but is not a total order.

Page 27 Section 1.6: added formal definitions of (1) “most-preferred strategy” and (2) “limit-plausible equilibrium”.

Pages 26-27 Section 1.6: added remark on the existence of a “most-preferred strategy” and a “limit-plausible equilibrium (what used to be Proposition 2).

Pages 27-28 Section 1.6: added a remark about what used to be Proposition 3 and its proof explaining why the limit-plausible equilibrium is a refinement of the Nash equilibrium.

Pages 28-29 Section 1.6: added a detailed example of a purely-aggregative game where a most-preferred strategy does not exist and therefore a limit-plausible equilibrium does not exist.

Page 29 Section 1.6: replaced the definition of an “overtaking best-response” with the definition of a “μ-dominant strategy” that reflects the fact the dominance relation is with respect to the given distribution of strategies: the μ distribution.

Page 29 Section 1.6: corrected and clarified what are now Proposition 2 and Corollary 1 to reflect the corrections in the related definitions. Also added a short remark on the motivation for these results.

Pages 29-30 Section 1.6: replaced the definition of an "overtaking α-dominated strategy" with the definition of an "α-μ-dominated strategy" that reflects the fact the dominance relation is with respect to the given distribution of strategies: the μ distribution.

Page 30 Section 1.6: corrected and clarified Proposition 3 to reflect the corrections in the related definitions. Also added a short remark on the motivation for this result and the related corollary.

Page 30 Section 1.6: added Corollary 2 (corresponding to Proposition 3) that reflects the fact that Proposition 4 applies also to "dominated strategies".

Pages 37-38 Appendix: corrected and clarified proofs of Propositions 2 and 3 to reflect the corrections in the related definitions and the definition of the "better-response overtaking preference relation" over strategies

Chapter 2

Page 41 Section 2.1: added remark clarifying motivation for the study of games with countably-many players

Page 43 Section 2.2: corrected the space of players to be an arbitrary countably-infinite, closed, and metrizable set

Page 44 Section 2.2: corrected the definition of a probability measure λ_N so that is not restricted to be equiprobable

Page 44 Section 2.2: corrected the definition of the “distributional form” of a player-set to include only the probability measure λ_N

Pages 45-46 Section 2.2.1: corrected the distributional form of sets B and K for double-counting of players

Pages 48-49 Section 2.4.1: corrected the metric space of firms for double-counting of players

Pages 52-53 Section 2.5.3: corrected the restricted Cournot games G_B and G_K for double-counting of players

Page 53 Section 2.6.1: corrected the space of all potential firms

Page 57 Section 2.7.1: corrected and clarified the definition of the “better-response preference relation”

Page 57 Section 2.7.1: clarified that the definition of the “better-response preference relation” is asymmetric and induced by a continuous payoff function

Page 57 Section 2.7.1 added remark on the relationship between the “better-response preference relation” and the Nash equilibrium concept.

Pages 60-61 Section 2.7.3: added the definition of a “limiting-game”

Pages 63-64 Section 2.8.2: corrected the example of distance between restricted games for double-counting of players

Page 67 Section 2.9.2: added Proposition 6 “completeness of the metric space of graphs of best-response correspondences in a complete metric space”

Pages 67-70 Section 2.10: added two subsections (2.10.1 and 2.10.2) discussing properties of strategic dominance and Nash equilibria of games in a complete metric space

Page 71 Appendix: corrected the proof of Proposition 4 to reflect the changes in the definition of “distributional form”

Chapter 3

Page 74 Section 3.1: added a remark clarifying that Harsanyi (1967) introduced player-types in an incomplete information setting while we use types in a complete information setting

Page 74 Section 3.1: added an example of a government imposing a tax-plan baed on types of tax-payers .This example clarifies the motivation for using types in a complete information setting

Page 75 Section 3.1: corrected the space of types and clarified that is an arbitrary countably-infinite closed and metrizable set

Page 78 Section 3.2: clarified the definition of the probability measure λ_N

Page 78 Section 3.2: corrected the definition of the “distributional form” of a type-profile to include only the probability measure λ_N

Pages 79-80 Section 3.2.1: corrected the example of firms with types for double-counting of types and the distribution of each type

Page 81 Section 3.3: corrected the example of a Cournot market game with types for double-counting of types and the distribution of each type

Contents

Acknowledgments

This thesis concludes my postgraduate and research studies at the Warwick Mathematics Institute. I had the privilege to complete two years of mathematics coursework and obtain an MSc in Interdisciplinary Mathematics with Distinction. This privilege was compounded by the opportunity to pursue a PhD under the direction of Professors Peter Hammond and Robert MacKay. During these years a number of special individuals inspired me and left a definite positive mark on my academic development. I am grateful to every one of them at an academic, and above all, personal level. In addition, financial support from the Engineering and Physical Sciences Research Council, and the Marie Curie Research Programme is gratefully acknowledged.

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Of course, I alone remain responsible for any errors or omissions which may unwittingly remain.

Abstract

Following Schmeidler (1973) and Mas-Colell (1984), economists have typically used aggregative games with a continuum of players to model strategic environments with a large number of participants. In these games a player's payoff depends on her own strategy and on an average of the strategies of everyone in the game. Examples include corporate competition in global markets, welfare maximization in multi-period economies, strategic voting in national elections, network congestion, and environmental models of pollution or, more generally, widespread externalities.

This study consists of three chapters. In Chapter 1 we unveil a weakness of the Schmeidler - Mas-Colell framework, and we develop a potential remedy that leaves the framework intact. In Chapter 2 we set the theoretical foundations for an alternative framework that is immune to the above weakness. Finally, in Chapter 3 we demonstrate how our approach accommodates types of players. We provide a number of fully worked-through examples and an appendix at the end of each chapter that includes the proofs to our propositions.

0.1 Summary

Consider the pair of a framework to model strategic environments, and a solution concept. In our case it is the framework of Schmeidler (1973) and Mas-Colell (1984), and the solution concept of Nash equilibrium. In Chapter 1 first we[1] prove that there is a mismatch between the framework and the solution concept. Second, we construct a solution concept that works well in the existing framework.

A game with infinitely-many players (either countably infinitely-many or uncountably-many players) can be viewed as a limit abstraction of the the concept of large games. We may study games with infinitely-many players but since we live in a world with finitely many people, we ultimately want to make predictions about the outcome of large finite-player games. Therefore, it is our view that the fitness of such a limit abstraction should be evaluated by examining how close are (1) its predictions of how the strategic situation with infinitely-many players will be resolved, to (2) the outcome of large games with finitely-many players.

In more detail, in Chapter 1 we argue for a new refinement of Nash equilibrium in purely-aggregative games with a continuum of players. First we prove that in the above class of games every strategy profile is a Nash equilibrium. We show that in many cases Nash equilibrium strategies of games with a continuum of players are *not* best-responses (i.e. they are suboptimal)

[1] A short note on personal pronouns. Regarding third-person singular pronouns, in this study my preference is to use "she" for all individuals. Quoting my teacher Martin J. Osborne: "*Obviously this usage is not gender neutral, but its use for a few decades, after a couple of centuries in which "he" has dominated, seems likely only to help to eliminate sexist ways of thought.*" (Osborne and Rubinstein (1994)). Regarding first-person pronouns, my preference is to use "we" as a cordial invitation to the reader in a short journey through the land of large games and their mathematics.

in any finite-player version of the game. This way we characterize the set of Nash equilibria that we consider to be implausible. Then we argue that this weakness is due to a side-effect of assuming a continuum of players. This side-effect renders players artificially indifferent among their strategies even though they are not indifferent among outcomes.

We define “Strategic Insignificance” and “Economic Negligibility” and explain why and how these two concepts differ. We also discuss why the framework of Schmeidler (1973) and Mas-Colell (1984) cannot distinguish between Strategic Insignificance and Economic Negligibility, and the side-effects of this. Then we attempt to restore Strategic Significance of players while maintaining their Economic Negligibility. We define “better-response preferences over strategies” and use them to construct our refinement of Nash equilibrium that we call “limit-plausible equilibrium”. Throughout this chapter we demonstrate our results and the superiority of our refinement over the regular Nash equilibrium through a number of examples. These examples include: welfare maximization in multi-period economies, competition of firms in a Cournot market, and strategic voting in elections with seats.

In Chapter 2 our method of treatment is the opposite of that of Chapter 1. We keep the solution concept (Nash equilibrium) unchanged and we change the framework. We set the theoretical underpinnings for an alternative framework for the study of large games, that does not give rise to implausible equilibria. We start virtually from scratch without depending on the existing economics literature and use only mathematics, new definitions and propositions. This is because the existing framework (Schmeidler (1973), Mas-Colell (1984)) is the only well-received approach in the economics literature up to now and as such permeates the related economics literature.[2]

Our aim is, at minimum, to provide a set of tools for the study of large games based on mainstream mathematical concepts like sequences, limits, and distributions. A set of tools that economists with basic knowledge of topology and measure theory can understand and find useful in studying game theory. Throughout Chapter 2 we follow closely an example of Cournot market competition that practically demonstrates our main constructs and results.

In Chapter 3 we apply our set of tools to strategic environments where participants have complete information and can be grouped into various "types". We demonstrate our results thoroughly with an example of Cournot market game that also allows direct comparison with the results of Chapter 2.

[2] There are three other approaches that are mathematically elegant but have been adopted so far only by a handful of economists: Khan and Sun (1999), Al-Najjar (2008), and Lasry, Lions, and Guant (2011). We mention in some detail these approaches also in Chapter 2.

Chapter 1

The Collective-Action Problem

1.1 Introduction

Games with a continuum of players were first introduced by Schmeidler (1973). One of main results of Schmeidler's paper is the first proof of pure-strategy Nash equilibrium existence in non-atomic games with a continuum of players. Schmeidler achieved this by restricting each player's payoff function to depend only on that player's own strategy and the integral of the strategies of all players. This integral can be seen as an average of the strategies of all players in the game. Since then, this restriction on the players' payoff function appears in virtually all models of games with a continuum of players, and it imposes additional structure on the channels through which players interact with each other. Players cannot affect directly each other's payoff except through the aggregate. It is the aggregate that specifies how and to what extent players can interact with each other. This class of games is now known as "non-atomic aggregative games".

In this study we follow the fundamental definition of Nash equilibrium as the strategy profile such that no player can obtain a higher payoff by unilaterally changing her strategy. We believe that this definition is particularly intuitive and speaks of the essence of Nash equilibrium. We also believe that, at Nash equilibrium, it is not intuitive to allow countably many players to have chosen strategies that are *not* best-responses irrespective of the other players' strategies.

We accommodate a number of standard conventions in economic theory such as the choice of $[0, 1]$ as the set of players, a compact space of actions, and other standards of the economics literature. We want this study to fit well into the relevant economics literature, appeal to game theorists, and provide insight into the analysis of games with infinitely many players.

Now we provide a straightforward example of a strategic situation that fits perfectly into the class of games with a continuum of players, and nonetheless, gives rise to equilibria that we consider to be implausible.

Elections have been declared! Consider a continuum of voters endowed with a non-atomic measure and a finite set of parties. Each party is allocated a finite number of parliamentary seats depending on the number of votes it receives. Assume that the more seats a party occupies, the better it can serve the interests of its supporters. We also assume that each voter's preferences are complete, strict and single-peaked, which are standard assumptions in models of strategic voting. By "complete" and "strict" we mean that, given any two election outcomes, every voter can choose which outcome she prefers more, and as a consequence of single-peakedness each voter has a globally most-preferred outcome.

Since voters belong to a continuum set endowed with a non-atomic measure, any single vote carries zero weight in the determination of the election outcome. This way any individual voter is rendered indifferent between voting for her most-preferred or least-preferred party or any other party. This indifference holds not just for a specific voting pattern of the other voters, but for *every* voting pattern of the other voters. Given *any* voting profile, no single voter has an incentive to change her vote, whatever that vote may be, and thus any strategy profile is a Nash equilibrium. Considering that a non-atomic game with a continuum of players has a continuum of strategy profiles, our voting game has infinitely many Nash equilibria. Note that by assumption voters are not indifferent with respect to the *outcome* of the elections; they are just indifferent between any two of their own strategies. For this reason we consider all equilibria, except the one where every player votes for her most-preferable party, to be implausible. Note that, in any finite version of these elections, a voter will always choose to vote for her most-preferred party, no matter what the other players have chosen to vote. This is because we have assumed that the more votes a party receives, the more parliamentary seats it is assigned, and the better it can serve the interests of its voters.

The rest of Chapter 1 is organized as follows. First in Section 1 we define "non-atomic purely-aggregative games" and prove that in this class of games every strategy profile is Nash equilibrium. In Section 2 we define "games of restricted participation", "restricted strategy profiles" and "restricted distributions of strategies". Through a number of examples we characterize the set of implausible Nash equilibria of a non-atomic game with a continuum of players. In Section 3 we discuss the concepts of Economic Negligibility and Strategic Insignificance, and take on the challenge of restoring Strategic Significance and interaction between players without negating Economic Negligibility. In Section 4 we define the "better-response preference relation over strategies" and the associated equilibrium-concept that we call "limit-plausible equilibrium". In addition we establish two results that are particularly useful in qualifying strategy profiles as limit-plausible equilibria, and in identifying implausible equilibria. Then we demonstrate the extent that our results facilitate analysis of three strategic environments: voter behavior in elections with seats, welfare maximization in a multi-period economy, and production optimization in a Cournot market. The proofs to our propositions are included in the appendix at the end of this chapter.

1.2 Related Literature

In the economics literature there is little discussion on the relation between Nash equilibria of games with a continuum of players, and Nash equilibria of games with finitely many players. On the one hand, researchers have suggested modifications to finite games in order to exhibit equilibrium properties of games with a continuum of players. For example, in the class of finite aggregative games, Alos-Ferrer and Ania (2005) define an optimal Aggregate-Taking Strategy as the strategy that, given the aggregate value that results when all players adopt it, is optimal for the individual. Under the assumption of Aggregate-Taking Strategies a player chooses her strategy without taking into account her impact on the aggregate exactly because the value of aggregate can be taken for granted. Consequently, even if this player has a non-negligible impact on the aggregate, she will behave as if her impact was indeed negligible. This nullifies strategic interaction even when the number of players is finite, and gives a result diametrically opposed to what we aim to achieve.

On the other hand, researchers have studied extensively the conditions under which equilibria of increasing sequences of economies are also equilibria of the corresponding non-atomic economy. The question, however, that we seek to address is more related to lower hemicontinuity[1]: we wish to *restrict* the set of Nash equilibria of games with a continuum of players to the equilibria that we consider to be plausible. We achieve this on the one hand by qualifying all plausible equilibria, and on the other hand by excluding any equilibrium involving strategies that are not best-responses in almost any finite-player version of the game.

Regarding lower hemicontinuity of Nash equilibrium, Khan and Sun (1999), Al-Najjar (2008), and Carmona and Podczeck (2009) establish results regarding *approximate* equilibria of games with finitely many players. They establish an equivalence relation between exact equilibria of non-atomic games and limiting ϵ-equilibria of finite games. An ϵ-equilibrium or "near-Nash equilibrium", is a strategy profile that satisfies the condition of Nash equilibrium *approximately*. The Nash equilibrium condition is not satisfied exactly because at least one player has chosen her strategies assuming (correctly) that her payoff has changed by $\epsilon > 0$. Our work differs from the work of Khan and Sun (1999), Al-Najjar (2008), and Carmona and Podczeck (2009) substantially in principle since we look at *exact* Nash equilibria, i.e strategy profiles that satisfy exactly the conditions of the original Nash equilibrium concept. We demand that these equilibrium conditions are satisfied both in the non-atomic game with a continuum of players and in the game with finitely many players.

[1]A correspondence $T : A \to B$ is said to be lower hemicontinuous at the point y if for any open set O such that $O \cap T(y) \neq \emptyset$ there exists neighborhood U of x such that $O \cap T(x) \neq \emptyset$ for all $x \in U$. A correspondence $T : A \to B$ is said to be upper hemicontinuous at the point y if for any open neighborhood O of T(y) there exists a neighborhood U of a such that for all $x \in U$, $T(x) \subset O$.

Most relevant to our objective are two working papers by Carmona (2004), and Barlo and Carmona (2011). In both papers the authors recognize that implausible Nash equilibria can exist in games with a continuum of players, and propose Nash equilibrium refinements.

Carmona (2004) proposes a Nash equilibrium refinement based on a limit concept. A "limit-equilibrium" of a non-atomic game with a continuum of players is the limit of a sequence of Nash equilibria when the sequence of corresponding finite games converges to the non-atomic game with a continuum of players. At first glance this approach may seem remarkably close to ours. Nonetheless it requires the simultaneous convergence of *entire* Nash equilibrium strategy profiles. This means that each player's strategy has to be in the same sequence with all other players' strategies. Our approach does not require the convergence of whole Nash equilibrium strategy profiles, but instead, the convergence of just singleton sequences of "most-preferred strategies". Each player can have her own, possibly distinct, sequence of "most-preferred strategies" that converges to a Nash equilibrium of the limit-game.

Barlo and Carmona (2011) define a strategic equilibrium as the limit of a sequence of equilibria of ϵ-perturbed games. In each perturbed game each player believes that *only* her action has an ϵ effect on the aggregate; the limit is attained as this belief tends to zero. Nevertheless this approach is based on the assumption that players are *not* rational. Barlo and Carmona admit that *"...in the ϵ-perturbed game agents are **not** rational, because an agent thinks that he alone has an ϵ impact on the societal choice, and does not foresee that other players have the same consideration as well"*. On the contrary, our refinement preserves individual rationality by letting players choose strategies as if they were playing a game where everyone has the same non-zero effect on the aggregate and this is common knowledge. The assumption of individual rationality permeates economic analysis and game theory, and violating it would create a number of paradoxes.

In Stergianopoulos (2008) we presented our first attempt to resolve the issue of implausible Nash equilibria in purely-aggregative games with a continuum of players. In the first part of that study we identified the issue of proliferation and implausibility of Nash equilibria, and proved existence of implausible Nash equilibria in a specific class of purely-aggregative games with a continuum of players.[2] In the second part of Stergianopoulos (2008) we proposed an equilibrium concept, "overtaking Nash equilibrium", based on overtaking preferences. The "overtaking Nash equilibrium" was *not* constructed following the approach based on better-responses that we follow in this study (see Section 1.6). As a result, in Stergianopoulos (2008) we were able to demonstrate existence of an "overtaking Nash equilibrium" equilibrium only in a specific subclass of purely-aggregative games where each player has a dominant[3] strategy.

Although this study shares the same point of departure and some of the basic principles with Stergianopoulos (2008), the scope of analysis is different. Our present study applies to a larger class of games since we need only a strict subset of the previous assumptions.[4] Second, our 2008 study possessed limited theoretical foundations. Here we employ more sophisticated tools and we set the theoretical underpinnings both for Stergianopoulos (2008) and for a new

[2] This class of games involved games with finitely many types of players and differentiable payoff functions.

[3] A dominant strategy is a strategy that results in a higher payoff for the player who employs it, irrespective of what the other players do. For example, in the classical prisoner's dilemma, betraying the trust of the other player is a dominant strategy for both players.

[4] For example, we do not require finitely many types of players or differentiable payoff functions.

framework for the study of games with many players.

1.3 Purely-aggregative Games

The main characteristic of purely-aggregative games is the structure of strategic interdependence among players: each player's payoff depends on her own strategy only through "collective action" which is signified by some real-valued function of all players' strategies in the game. The fact that payoffs may be determined solely by collective action has been pointed out among others by Jensen (2008), Rath (1992), and Vives (1999). In addition, Jensen (2008) and Rauh (1997) suggested that collective action can be modeled as a statistic, like the mean of all players' strategies or even a higher moment. In the class of purely-aggregative games belong many applications of game theory that are of substantial interest to economists: elections, environmental models of pollution, models of managing common resources, network congestion, market competition, public goods, and generally, all widespread externalities (Kaneko and Wooders (1994)). In general, the class of purely-aggregative games involves any strategic situation where individuals are affected predominantly by an aggregate of the behavior of others, and where permuting the strategies of any number of players has no effect on the payoff of the rest of the players since its the collective action that matters, and not specifically who plays what.

1.3.1 Games with a continuum of players

Let $\mathcal{I} := [0,1]$ denote the set of players where a player is a real number in that interval, and λ denote the Lebesgue measure on $\mathcal{I}$. The Lebesgue measure is an example of a non-atomic probability measure. Such a measure does not admit atoms, or equivalently it requires that every set of positive measure has a proper subset of positive measure. The related sigma-algebra is denoted by $\mathbb{B}$.

The set of actions for each player $i \in \mathcal{I}$ is denoted by A which is a compact space, and $\Delta(A)$ denotes the set of all Borel probability measures on A. We interpret a measure $\mu \in \Delta(A)$ as the proportion of players whose strategies belong to the same set, i.e.

$$\mu(B \subset A) = \lambda(\{i \in \mathcal{I} \mid \alpha^i \in B\}$$

where α^i is the strategy of player i.

It is worth noting that we consider exclusively single-move games. In these games the main distinction between an action and a strategy is that a strategy is the one action that is played.[5] We will use the term "strategy" whenever the two terms are equivalent, and make a clear distinction between strategies and actions whenever risk of confusion arises.

Definition 1. *A non-atomic game with a continuum of players denoted by*

$$G := \langle \{\mathcal{I}, \mathbb{B}, \lambda\}, A, p \rangle$$

is said to be ***purely-aggregative*** *if the payoff of each player depends only on the distribution of strategies of all players. Mathematically, if for every player* $i \in \mathcal{I}$ *we can express her payoff function* p^i*, as*

$$p^i : \Delta(A) \to \mathbb{R}$$

[5]It is in multiple-move games that a strategy in conceptually different from an action. In these games, strategies involve a list of actions played at each stage of the game.

Note that we assume p^i to be continuous and that λ is a non-atomic probability measure and thus, any single player has zero measure.[6] Sets of zero measure do not affect the distribution and consequently the strategy of any single player cannot affect the distribution of strategies of all players in the game.

We believe that conceptually the Schmeidler - Mass-Colell framework includes the class of purely aggregative games. Schmeidler (1973), on the one hand, states that "*Non-atomic games enable us to analyze a conflict situation where the single player has no influence on the situation but the aggregative behavior of "large" sets of players can change the payoffs*". On the other hand, in the mathematical model each player's payoff is a function of both her own strategy and of the distribution of strategies of all players. One could discard Schmeidler's (1973) framework as inappropriate to model purely-aggregative games with a continuum of players simply on that account. We wish to provide an additional reason in support of the need for an alternative framework for the study of games with infinitely-many players. That is precisely why we give a second chance to the Schmeidler - Mass-Colell framework (or "the benefit of the doubt") and through Proposition 1 reveal a weakness of this framework in the class of purely-aggregative games with a continuum of players. With the following proposition we establish that in the aforementioned class of games the only prediction one can make using the Nash equilibrium is that every outcome is possible. Note that even in *any* finite version of the game there

[6] Here we provide a short proof that when players belong to a set $\mathcal{I}$ endowed with a non-atomic measure λ, single players have zero weight. We claim no originality of this proof but we do not have a reference for it. A measure λ on $\mathcal{B}$ is non-atomic $\Leftrightarrow$ for every $B \in \mathcal{B}$ such that $\lambda(B) > 0$ there exists $K \subset B$ such that $\lambda(B) > \lambda(K) > 0$. Since the only subset of a singleton set is the empty set and $\lambda(\emptyset) = 0$, it should be the case that $\lambda(\{i\}) = 0$ for every $i \in \mathcal{I}$.

are strategies profiles that are not Nash equilibrium, in the continuum version of the game every strategy profile is a Nash equilibrium In that sense, Nash equilibrium is a vacuous solution concept of purely aggregative games with a continuum of players because it does not "solve" the game in a non-trivial way. We want to emphasize that by definition, the only way that *every* strategy profile is Nash equilibrium of a game is if *every* strategy is a best-response. This means that the best-response correspondence is the entire strategy set of each player, and using Nash equilibrium we cannot make any useful prediction of how the game will be played.

Proposition 1. *In a purely-aggregative non-atomic game with a continuum of players G every strategy profile is a Nash equilibrium.*

A proof is provided in the Appendix.

We understand that a reader knowledgeable in game theory would could consider the proof of Proposition 1 immediate, and indeed the proof follows directly from the related definitions. We include Proposition 1 and its proof for completeness and we believe that its value does not lie in establishing a ground-breaking result. Our aim is to make available to the reader who is less accustomed to games with infinitely many players a proof that makes clear how the related concepts interact with each other and the exact point of departure for our analysis in this chapter.

Note that although that Proposition 1 applies only to purely-aggregative games, our examples of a Cournot market game and of welfare maximization in a multi-period economy are general aggregative games. As we will see in Section 1.5, in these examples not every strategy profile is a Nash equilibrium of the non-atomic game with a continuum of players but many Nash equilibria can be considered implausible since they could not exist in any finite version of the game.

1.3.2 Games of restricted participation

In this section we define games of restricted participation and we link games with a continuum of players to games with a finite set of players. We consider finite sets of players that are subsets of the given continuum set of players. A "restricted game" is the restriction of the non-atomic game with a continuum of players to a finite number of players. The purpose of the restricted game is to simulate the strategic environment that would arise if participation in the non-atomic game with a continuum of players was restricted to a finite subset of players. Furthermore, we want to exclude from our study non-atomic games that in their restricted games *every* strategy profile is a Nash equilibrium. In other words, we want to consider non-atomic games whose restricted games have strategy profiles that are Nash equilibria, and strategy profiles that are *not* Nash equilibria. This way we will have some idea of what outcome of the game could be considered plausible based on Nash equilibrium criterion, and what outcome could not be considered plausible.[7]

If we endow $\Delta(\mathcal{I})$ with the topology of weak convergence of measures [8], and denote the associated Prohorov metric by d_P then $(\Delta(\mathcal{I}), d_P)$ is a compact space like $\mathcal{I}$ (Theorem 15.15 in Aliprantis and Border (2005)). For each natural

[7]Our analysis and our equilibrium concept do apply also to non-atomic games with restricted games where every strategy profile is Nash equilibrium. Be that as it may, we exclude these games because we believe that they are not interesting.

[8]A sequence $\{\lambda_N\}$ of measures converges "weakly" (or "in distribution") to λ, and we can write $\lambda_N \Rightarrow \lambda$, if

$$\lim_{\#N\to\infty} \int_{\mathcal{I}} f(i) d\lambda_N(i) = \int_{\mathcal{I}} f(i) d\lambda(i)$$

for every continuous, bounded, real-valued function $f : \mathcal{I} \to \mathbb{R}$.

number $n \in \mathbb{N}$, let $\Delta(\mathcal{I}) \subset \Delta(\mathcal{I})$ consist of those probability measures such that

$$\lambda_N(\{i\}) \in \{0, \frac{1}{\#N}\} \text{ for each } i \in \mathcal{I}$$

Then for a set $N \subset \mathcal{I}$ we have

$$\lambda_N(\{i\}) = \frac{1}{\#N} \text{ if } i \in N, \text{ and } \lambda_N(\{i\}) = 0 \text{ if } i \in \mathcal{I} \setminus N$$

which means that player i participates in the game with $\#N$ players and has weight $\frac{1}{\#N}$. If player $i \in \mathcal{I}$ does not participate in that game then $\lambda_N(\{i\}) = 0$. The measure λ_N selects from the space of all potential players, the $\#N$ players that actually participate in the game.

Let $\mathcal{F}(\mathcal{I}) \subset 2^{\mathcal{I}}$ denote the family of all finite subsets of $\mathcal{I}$.

Definition 2. *Given a non-atomic game with a continuum of players G, the* ***restricted game*** *with player-set $N \in \mathcal{F}(\mathcal{I})$ is a triple*

$$G_N := \langle \{N, \lambda_N\}, A, p_N \rangle$$

where

$$\lambda_N(\{i\}) := \frac{1}{\#N} \text{ for every } i \in N$$

is the uniform probability measure over $\#N$ objects. The payoff function

$$p_N^i : \Delta(A) \to \mathbb{R} \text{ for } i \in N$$

is the restriction of p^i to the set of distributions Δ with $\#N$ equiprobable atoms, and is assumed to be continuous.

Definition 3. *Given a strategy profile $\mathcal{A}$ of a non-atomic game with a continuum of players G, the* ***restricted strategy profile*** $\mathcal{A}_N$ *of the game G_N is the restriction of $\mathcal{A}$ to the player-set $N \in \mathcal{F}(\mathcal{I})$. Also, given a distribution $\mu \in \Delta(A)$ of strategies of all players in the non-atomic game G, the* ***restricted distribution of strategies*** $\mu_N \in \Delta(A)$ *of the game G_N is the restriction of μ to the player-set $N \in \mathcal{F}(\mathcal{I})$.*

As we mentioned at the beginning of this subsection, it is important that we consider games where the Nash equilibrium criterion excludes some outcomes. To do this we consider restricted games where not every strategy profile is a Nash equilibrium.

Definition 4. *A restricted game G_N with finitely many players is* ***non-degenerate*** *if not every strategy profile is a Nash equilibrium of G_N. Similarly, a game G with a continuum of players is* ***non-degenerate*** *if for arbitrarily large but finite $N \in \mathcal{F}(\mathcal{I})$ the restricted game G_N is non-degenerate.*

1.4 Economic Negligibility and Strategic Insignificance

In this section we study the concept of individual negligibility in economics. In economic theory individuals are considered to be negligible when they have a negligible effect on aggregates, for example prices, inflation, and total production. We make a conceptual distinction with practical consequences to our analysis. We split the concept of individual negligibility into "economic negligibility" and "strategic insignificance". These two concepts are similar though distinctively different but in the continuum framework these two concepts coincide. This coincidence constitutes a serious weakness of the continuum framework and we demonstrate this through a number of examples. The dichotomy we make reveals that formulating strategic environments with many participants as games with a continuum of players has a dual effect on the strategic idiosyncracy of individuals. On the one hand it ensures that individuals cannot manipulate aggregates, and on the other hand it robs individuals of any strategic control over their own payoff. Most importantly, this separation guides us to formulate an equilibrium concept that remedies the weakness of Nash equilibrium in purely-aggregative non-atomic games.

Individual economic negligibility is a fundamental property that models of perfect competition need either to possess endogenously or to assume exogenously.

Definition 5. ***Economic negligibility*** *describes the situation where no single player can affect the value of an aggregate that affects players' payoff and is defined with respect to the functional form of the aggregate and the specific strategic situation that the players participate in.*

Such an aggregate can be: price, total quantity, total traffic or emissions, or any kind of average of the strategies of all players. For example, in an economy with a continuum of agents economic negligibility is endogenous since every agent carries zero weight and thus cannot affect prices. The very moment that agents have any power over prices we cannot regard the respective markets as perfectly competitive nor the associate models as representative of competitive markets.[9]

[9] Khan (2008) offers a more extensive discussion of the relationship between economic negligibility and perfect competition.

A seemingly equivalent concept, strategic insignificance, is a consequence of assuming uncountably many individuals. This is the case when individuals are represented as elements of a set with the cardinality of the continuum. In a wide class of games, strategic insignificance deprives players of any control over the payoff outcome.

Definition 6. ***Strategic insignificance*** *describes the situation where a player with well-ordered preferences over outcomes is rendered indifferent between any two of her strategies because neither strategy can affect the outcome, independently of what the other players choose to do.*

Strategic insignificance creates a situation where any strategy profile is Nash equilibrium. Note that at Nash equilibrium players are allowed to be indifferent between some of their strategies and this is what we call "Nash-equilibrium indifference". At Nash equilibrium no player should have an incentive to deviate from her chosen strategy *given* the *specific* response of the other players. Nevertheless, in games where not every strategy profile is Nash equilibrium (non-degenerate games), this indifference cannot hold for *all* possible responses of the other players; that would violate rationality.

Before we proceed to examples we should emphasize that the concepts of economic negligibility and strategic insignificance coincide in large class of game-theoretic models with a continuum of players. This class includes purely-aggregative games with a continuum of players.

Basically, the two concepts will coincide in *any* game where both the following two conditions hold:

a) every player's payoff depends entirely on an aggregate such as price, total quantity, total traffic or emissions, or any kind of average of the strategies of all players

b) no single player has the power to manipulate this aggregate

1.4.1 Example: Economic Negligibility without Strategic Insignificance

Consider a non-atomic game with a continuum of players where each player's payoff depends entirely on that player's cost of exerting effort. Each player's effort has zero weight in the determination of the total effort and thus each player is economically negligible. Nonetheless, each player can affect her own level of effort and thus her own payoff. This way players are not necessarily indifferent between their strategies and thus not strategically insignificant.

1.4.2 Example: Strategic Insignificance without Economic Negligibility

Typically, when players can affect aggregates then they are able to affect also their own payoff. In games with a continuum of players, economic negligibility holds precisely because that is the modeler's intent. Even so, if we slightly bend our modeling rationality and stretch our assumptions, we can construct game-theoretic scenarios where players are strategically insignificant without being economically negligible.

Consider a game where each player's payoff is determined solely by an action that some *other* player takes. For example, imagine a situation where

i. high noise coming from a neighbor's house has such a detrimental effect on a player's wellbeing that, above a level, it fully determines her payoff

ii. no player has control over her neighbor's actions but she can affect (e.g. increase) the total noise level in her neighborhood

Therefore strategic insignificance holds while economic negligibility does not.

1.5 Examples of Implausible Nash equilibria

In this section we examine some applications of game theory that are of substantial interest to economists. In each example we identify the implausible Nash equilibria and explain in detail why we consider them to be implausible. Then we compare the equilibrium behavior in the non-atomic game with a continuum of players with the equilibrium behavior in an arbitrary restricted game. In all cases the comparison of equilibrium behavior supports our choice of implausible Nash equilibria. We look at equilibrium strategies, as opposed to equilibrium distributions, because according to Carmona (2006) in games with countable action space all equilibrium behavior is captured by equilibrium strategies. It is worth noting that we do not claim that there is no plausible Nash equilibrium of the game G with a continuum of players. We rather claim that there exists at least one implausible Nash equilibrium of G.

A brief clarification regarding some of our notation: throughout this study we use the letter j as a superscript to denote the strategy of a specific player under consideration e.g. α^j, β^j, γ^j and we use the letter i in sums and integrals i.e. $\sum_{i\in N} q^i$, $\int_{i\in N} q^i d\lambda$ and also for players in general when we do not single-out a specific player.

1.5.1 Intertemporal welfare maximization

The following scenario is motivated by an example provided by Hammond (2007). There is a two-period economy G with a single, homogeneous, and countably divisible good, a continuum of agents endowed with a non-atomic probability measure λ, and a "benevolent" policy-maker. In the beginning of period 1 each agent is endowed a fixed quantity of the good. Then each agent decides how much to save for period 2 and consumes the remainder during period 1. In period 2 there are no decisions, just consumption of what has been saved in period 1. The consumption of agent $j \in \mathcal{I}$ in periods 1 and 2 is denoted by c_1^j and c_2^j respectively, and her initial endowment is denoted by e^j. We assume that it is *not* possible for each agent to consume her entire endowment in a single period without becoming satiated. That is, agents will reach their maximum utility and there still be available units for consumption. We do assume however that agents can dispose any quantity of the good at no cost.

Each agent seeks to maximize her utility over the two periods

$$u^j(c_1^j, c_2^j) := u^j(c_1^j) + u^j(c_2^j)$$

subject to the individual feasibility constraint

$$c_1^j + c_2^j \leq e^j$$

The policy-maker is benevolent in that she seeks to maximize a Bergson social welfare function of the form

$$U(c_1, c_2) := \int_{\mathcal{I}} u^i(c_1^i,\ c_2^i) d\lambda$$

subject to individual and aggregate feasibility constraints.[10] According to the example provided by Hammond (2007), in order to achieve the intertemporal welfare optimum, it is as if the policy-maker confiscates all savings after period 1 and redistributes them equally to all agents in period 2. Therefore each agent $j \in \mathcal{I}$ has expected utility

$$u^j(c_1^j, c_2^j) := u^j(c_1^j) + u^j(\int_{\mathcal{I}} (e^i - c_1^i) d\lambda)$$

where $\int_{\mathcal{I}} (e^i - c_1^i) d\lambda)$ is the average saving per head available in period 2.

Note that no single agent can affect $\int_{\mathcal{I}} (e^i - c_1^i) d\lambda$ and hence cannot affect her utility in period 2. It is not that agents do not care about their utility in period 2 but that there is no way for any single agent to affect it. In this economy with the benevolent policy-maker, agents can take their utility in period 2 as given and seek to maximize only their utility in period 1. Each agent knows that her own savings have negligible effect on the aggregate of the second period, and on that account period 2 is simply not relevant to her maximization problem, and need not be considered.

In this example the set of implausible Nash equilibria includes the following strategy profiles:

1) The strategy profile where every agent saves nothing for period 2

[10] These constraints guarantee that in period 1 no individual consumes more than her endowment, and that in period 2 total consumption equals total savings.

2) All strategy profiles where only countably many agents save something for period 2

In either case, this kind of behavior would lead to $\int_{\mathcal{I}}(e^i - c_1^i)d\lambda = 0$. This means that we would get zero consumption in period 2. We believe that it is reasonable to assume that zero consumption in any single period leads to death within that period. Analogously, zero consumption overall means death of all agents in the economy. With death being the least desirable outcome we can safely regard zero consumption as an implausible outcome.

Now we examine an arbitrary restricted economy G_N. In this economy the expected utility of any agent $j \in N$ is

$$u^j(c_1^j, c_2^j) := u^j(c_1^j) + u^j(\frac{1}{\#N}\sum_{i \in N}(e^i - c_1^i))$$

Individual savings can affect $\frac{1}{\#N}\sum_{i \in N}(e^i - c_1^i)$ since

$$\frac{\mathrm{d}\frac{1}{\#N}\sum_{i \in N}(e^i - c_1^i)}{\mathrm{d}(e^j - c_1^j)} = \frac{1}{\#N} > 0$$

for every $j \in N$ and on that account all agents will choose a strictly positive level of savings for period 2.

Here we see that while the outcome of "zero consumption in period 2" is a Nash equilibrium of the economy with a continuum of agents, it is *not* a Nash equilibrium of any economy with finitely-many agents.

1.5.2 Cournot market game

Consider a Cournot market game $G = \langle \mathcal{I}, \mathbb{B}, A, p \rangle$ with a continuum set of firms $\mathcal{I} := [0, 1]$ endowed with a non-atomic measure λ. For a single homogeneous good, each firm $j \in \mathcal{I}$ decides its level of production

$$q^j \in A^j := [0, q_{max}] \cap \mathbb{Q}$$

and $q_{max} \geq 1 - c$ where $c \in [0, 1)$ is the unit cost of production. The function $P(Q) = 1 - Q$ relates the price P of the good with the average quantity produced $Q := \int_{\mathcal{I}} q^i d\lambda$. Then the payoff function of firm j can be represented by its profit function

$$p^j(q^j, Q) := (1 - Q)q^j - cq^j = (1 - \int_{\mathcal{I}} q^i d\lambda - c)q^j$$

The Nash equilibrium strategies belong to the set

$$\{q^j \in A^j \ \mid 1 - \int_{\mathcal{I}} q^i d\lambda - c = 0\}$$

because only if $1 - \int_{\mathcal{I}} q^i d\lambda - c = 0$ does the profit of every firm equal zero and so there is no incentive for a firm to change its current production level. Therefore if $1 - \int_{\mathcal{I}} q^i d\lambda - c = 0$ then $p^j(q^j, Q) = 0$ for any $q^j \in [0, q_{max} \geq 1 - c] \cap \mathbb{Q}$ and for every firm $j \in \mathcal{I}$. Otherwise, if $1 - \int_{\mathcal{I}} q^i d\lambda - c \neq 0$, for some $q^j \in [0, q_{max} \geq 1 - c] \cap \mathbb{Q}$ we have

$$\frac{\partial p^j(q^j, Q)}{\partial q^j} \neq 0$$

and therefore there are firms that can affect their payoff and have a clear incentive to change their production level; we are not at Nash equilibrium.

We established that at Nash equilibrium $1 - \int_{\mathcal{I}} q^i d\lambda - c = 0$ should hold and the profits of every firm will be zero. Therefore firm j is anyway making zero profit, any level of production in $[0,\ q_{max}] \cap \mathbb{Q}$ can be part of a Nash equilibrium.

Now we examine an arbitrary restricted game G_N. In this game the average quantity is

$$Q := \frac{1}{\#N} \sum_{i \in N} q^i$$

Since

$$\frac{\partial Q}{\partial q^j} = \frac{1}{\#N} > 0$$

for every $j \in N$ we have

$$\frac{\partial p^j(q^j, Q)}{\partial q^j} = 1 - c - \frac{1}{\#N}(\sum_{i \in N} q^i + q^j) = 0$$

for

$$q^{*j} = \frac{1-c}{\#N+1} \in (0, q_{max}] \cap \mathbb{Q}$$

Consequently there is a unique symmetric Nash equilibrium where each firm maximizes its profits by producing its optimal quantity q^*.

Here we see that while any $q^j \in [0,\ q_{max}] \cap \mathbb{Q}$ is a Nash equilibrium of the non-atomic market game with a continuum of players, not every q^j is part of a Nash equilibrium of a market game with finitely-many players.

1.6 Limit-plausible equilibrium

In the examples of the previous section we demonstrated that none of the implausible Nash equilibria could exist in any finite version of the non-atomic game with a continuum of players. Nevertheless, this is not due to the "finiteness" property per se, but due to the non-negligible impact that each player has on the payoff outcome. This non-negligible impact on the aggregate creates an incentive for each player to choose a strategy that increases her payoff. Even the slightest impact of an individual's strategy on the aggregate, suffices to break the artificial strategic insignificance.

Now we encounter the following paradox: on the one hand, from the point of view of models of perfect competition, it is desirable that single players are not able to manipulate aggregate statistics e.g. demand, supply, prices. On the other hand, from the point of view of game theory, it is desirable that single players choose strategies as if they could affect those aggregates. This would mean that players can interact with each other even if this happens only via the aggregate. In this case we would have economically negligible players that are strategically significant and do think strategically in their interaction. Our solution to this paradox is to endow players with "better-response overtaking preferences" over strategies. It is important to emphasize that we do not need any individual's strategy to actually have an impact on the aggregate, but we do need players to behave *as if* their strategies have such an impact. A real-world example that people behave as if they have some effect on the aggregate is national elections. Many people do vote even though in national elections any single voter is, in essence, economically negligible. Our concept of overtaking preferences is new to the literature but has its game-theoretic origins in Rubinstein (1979).[11] Rubinstein used the overtaking criterion to distinguish between two sequences of payoffs that have the same limit but differ in a large finite interval. In our context the overtaking property implies that from a finite point and beyond a strategy α is always better than

[11]In welfare economics the idea of the overtaking criterion is due to Hammond (1975) and in optimal growth theory due to von Weizsäcker (1965).

a strategy β. In other words, if strategy α is "better" in the overtaking sense than strategy β then this will also hold in every game with more players. Below we provide precise definitions of the related concepts.

Definition 7. *For player $j \in N \in \mathcal{F}(\mathcal{I})$ strategy $\alpha \in A$ is a* ***better-response*** *than strategy $\beta \in A$ given the distribution of strategies $\mu_{N\setminus\{j\}}$ if*

$$p^j(\alpha, \mu_{N\setminus\{j\}}) \geq p^j(\beta, \mu_{N\setminus\{j\}})$$

With the following definition we state that choosing between two strategies a player will prefer the strategy that gives her a higher payoff given what the other players have chosen to do, or be indifferent between the two strategies if they give her equal payoff (given what the other players have chosen to do).

Definition 8. *Given a restricted game G_N, the* ***better-response preference relation*** *over strategies of player $j \in N$ is a family of binary relations*

$$\succsim^j \quad \text{on} \quad A \times A \quad \text{and indexed by} \quad \Delta_{\#N-1}(A)$$

that satisfies the following requirements:

(1) $\alpha \succ^j_{N,\mu_{N\setminus\{j\}}} \beta \quad \text{if} \quad p^j(\alpha, \mu_{N\setminus\{j\}}) > p^j(\beta, \mu_{N\setminus\{j\}})$

and this means that $\alpha \succsim^j_{N,\mu_{N\setminus\{j\}}} \beta$ *but not* $\beta \succsim^j_{N,\mu_{N\setminus\{j\}}} \alpha$

(2) $\alpha \sim^j_{N,\mu_{N\setminus\{j\}}} \beta \quad \text{if} \quad p^j(\alpha, \mu_{N\setminus\{j\}}) = p^j(\beta, \mu_{N\setminus\{j\}})$

and this means that $\alpha \succsim^j_{N,\mu_{N\setminus\{j\}}} \beta$ *and also* $\beta \succsim^j_{N,\mu_{N\setminus\{j\}}} \alpha$

The first requirement states that in the finite game with player-set N, player j will ***prefer*** *strategy α over strategy β if strategy α provides her with a higher payoff than strategy β given the distribution $\mu_{N\setminus\{j\}}$ of strategies of all other players.*

The second requirement states that in the finite game with player-set N, player j will be ***indifferent*** *between strategy α and strategy β if strategy α provides her with the same payoff as strategy β given the distribution $\mu_{N\setminus\{j\}}$ of strategies of all other players.*

Note that $\succ^j_{N,\mu_{N\setminus\{j\}}}$ is asymmetric since $\alpha \succ^j_{N,\mu_{N\setminus\{j\}}} \beta$ means that $p^j(\alpha, \mu_{N\setminus\{j\}}) > p^j(\beta, \mu_{N\setminus\{j\}})$ but not $p^j(\beta, \mu_{N\setminus\{j\}}) > p^j(\alpha, \mu_{N\setminus\{j\}})$. Also $\succcurlyeq^j_{N,\mu_{N\setminus\{j\}}}$ is a total order and complete on the compact space A since it is induced by a continuous payoff function p^j_N.

With the following definition we declare that in a non-atomic game with a continuum of players, given a distribution of strategies μ, a player will prefer strategy α over strategy β if she also prefers α over β given *any* distribution of strategies in a sequence $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ that converges to μ.

Definition 9. *Given a non-atomic game with a continuum of players G, the* ***better-response overtaking preference relation*** *over strategies of player $j \in \mathcal{I}$ is a family of binary relations*

$$\succcurlyeq^j_* \quad on \quad A \times A \text{ and indexed by } \Delta(A)$$

that satisfies the following requirements for $\alpha \succ^j_{*,\mu} \beta$ and for $\alpha \sim^j_{*,\mu} \beta$.

$\boldsymbol{\alpha \succ^j_{*,\mu} \beta}$ *if there exist:*

(a) a finite number of players m

(b) an expanding sequence[12] *of player-sets $\{N\}_{\#N \geq m}$ with $j \in N$*

such that for any *sequence $\{\mu_{N\setminus\{j\}}\}_{\#N \geq m}$ of distributions of strategies of all other players that converges to μ it holds that $\boldsymbol{\alpha \succ^{j}_{N,\mu_{N\setminus\{j\}}} \beta}$*
Similarly for $\alpha \sim^{j}_{*,\mu} \beta$

Note that $\succ^{j}_{*,\mu}$ is asymmetric since $\alpha \succ^{j}_{*,\mu} \beta$ means $\alpha \succ^{j}_{N,\mu_{N\setminus\{j\}}} \beta$ (for any distribution $\mu_{N\setminus\{j\}}$ with $\#N \geq m$) but not $\beta \succ^{j}_{N,\mu_{N\setminus\{j\}}} \alpha$ (for any distribution $\mu_{N\setminus\{j\}}$ with $\#N \geq m$) and hence not $\beta \succ^{j}_{*,\mu} \alpha$.

In summary, in the game G, player j will prefer strategy α over strategy β if there exists a finite number of players m such that strategy α provides her with a higher payoff than strategy β, given *any* distribution of strategies $\mu_{N\setminus\{j\}}$ with player-set N, in *any* sequence $\{\mu_{N\setminus\{j\}}\}_{\#N \geq m}$ that converges to μ. Similarly, in the game G, player j will be indifferent between strategy α and strategy β if there exists a finite number of players m such that strategy α provides her with the same payoff as strategy β, given *any* distribution of strategies $\mu_{N\setminus\{j\}}$ with with player-set N, in *any* sequence $\{\mu_{N\setminus\{j\}}\}_{\#N \geq m}$ that converges to μ.

Definition 10. *For a player $j \in \mathcal{I}$ a* ***most-preferred strategy****, given a distribution of strategies μ, is a strategy α such that $\alpha \succcurlyeq^{j}_{*,\mu} \beta$ for every $\beta \in A$.*

Note that $\succ^{j}_{*,\mu}$ is *not* a total order and therefore existence of a "most-preferred strategy" cannot be guaranteed.

[12] By "expanding sequence" we mean that each player-set is included in the next player-set in the sequence i.e. $N_k \subset N_{k+1}$ for every $k \in \mathbb{N}$

Definition 11. *A strategy profile $\mathcal{A}$ is said to be a* ***limit-plausible equilibrium*** *of G if every player chooses a most-preferred strategy.*

Note that the limit-plausible equilibrium is indeed a refinement of the Nash equilibrium in purely-aggregative games with a continuum of players since we have shown that every strategy profile of G is a Nash equilibrium of G.

Here we provide a generic example of a purely-aggregative game in which a most-preferred strategy does not exist for at least one player, and therefore, a limit-plausible equilibrium does not exist.

Consider a non-atomic game with a continuum of players $G = \langle(\mathcal{I}, \mathbb{B}, \lambda), A, p\rangle$ where each player $j \in \mathcal{I}$ chooses a rational number $\alpha^j \in \mathbb{Q} =: A$ as her strategy, and her payoff is a function exclusively of the average $\bar{\alpha}$ of the strategies of all players in the game (or equivalently the distribution of strategies of all players μ). We assume that *not* all players have same payoff functions.

In the restricted game $G_N = \langle(N, \lambda), A, p_N\rangle$ assume that for player j strategy α^j is a better-response than strategy β^j given the average of the strategies of all other players $\bar{\alpha}_{N\setminus\{j\}}$ (or equivalently the distribution of strategies of all other players $\mu_{N\setminus\{j\}}$). Also, without loss of generality, assume that $\alpha^j > \beta^j$ (since all strategies in this game can be numerically compared because they are numbers).

As the number of players increases and more players enter the game, a new average $\bar{\alpha}_{M\setminus\{j\}}$ with $N \subset M$ will be determined (and equivalently a new distribution of strategies of all other players $\mu_{M\setminus\{j\}}$). For player j, strategy α^j will be a better-response than strategy β^j given the new average $\bar{\alpha}_{M\setminus\{j\}}$ only if the new players choose strategies in a specific way that results to an average that still is smaller than the average that maximizes her payoff.

Therefore, along most sequences of distributions of strategies $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$, what is a better-response strategy for player j will change at least once and the better-response overtaking preference relation is *not* well-defined. We see that in this example, a most-preferred strategy does *not* exist for at least one player (player j), and therefore, a limit-plausible equilibrium does *not* exist.

Even though a limit-plausible equilibrium may not always exist, in Section 1.7 we study examples of three important strategic environments where a limit-plausible equilibrium does exist.

Definition 12. *A strategy is* ***μ-dominant*** *if it provides at least as high payoff to player j as any other strategy given $\mu_{N\setminus\{j\}}$ and for any sequence $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ of distributions of strategies of all other players that converges to μ.*

Note that a μ-dominant strategy is not necessarily a "dominant strategy". This is because a μ-dominant strategy provides to j at least as high payoff as any other strategy, *only* for $\mu_{N\setminus\{j\}}$ or for sequences $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ that converge to μ, but *not* for *any* distribution of strategies of all other players. On the contrary, a dominant strategy is necessarily a μ-dominant strategy since it provides to j higher payoff than any other strategy, given *any* distribution of strategies of the other players.

With the following proposition we establish that the limit-plausible equilibrium qualifies "μ-dominant" strategies (as well as "dominant" strategies in the corollary).

Proposition 2. *Let G be a purely-aggregative non-atomic game with a continuum of players, such that in an arbitrarily large restricted game G_N, for player*

*$j \in N$ there exists a μ-**dominant** strategy α^j. Then in a limit-plausible equilibrium of G (with induced distribution of strategies μ) player j will choose strategy α^j (or an equivalent strategy).*

A proof is provided in the Appendix.

Corollary 1. *Let G be a purely-aggregative non-atomic game with a continuum of players, such that in an arbitrarily large restricted game G_N, for player $j \in N$ there exists a **dominant** strategy α^j. Then in a limit-plausible equilibrium of G (with induced distribution of strategies μ) player j will choose strategy α^j (or an equivalent strategy).*

Definition 13. *A strategy β^j is α^j_μ-**dominated** if it provides lower payoff to player j than the strategy α^j given $\mu_{N\setminus\{j\}}$ and for any sequence $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ of distributions of strategies of all other players that converges to μ.*

Note that an α^j_μ-dominated strategy is not necessarily a "dominated strategy".[13] This happens for two reasons: first because an α^j_μ-dominated strategy is not dominated in given *any* distribution of strategies of all other players, but only given the *given* $\mu_{N\setminus\{j\}}$ and for any sequence $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ of distributions of strategies of all other players that converges to μ. On the contrary, a dominated strategy is necessarily an α^j_μ-dominated strategy since it is dominated given *any* distribution of strategies of the other players.

With the following proposition we establish that the limit-plausible equilibrium does not qualify "α^j_μ-dominated" strategies (as well as any "dominated" strategy in the corollary).

[13] A "dominated" strategy should not to be confused with a "dominant" strategy. If strategy β^j is dominated by the strategy α^j, then β^j provides to player j lower payoff than α^j, *irrespective* of what the other players do.

Proposition 3. *Let G be a purely-aggregative non-atomic game with a continuum of players. A limit-plausible equilibrium of G (with induced distribution of strategies μ) cannot involve strategies that are α^j_μ-**dominated** for some player $j \in \mathcal{I}$.*

A proof is provided in the Appendix.

Corollary 2. *Let G be a purely-aggregative non-atomic game with a continuum of players. A limit-plausible equilibrium of G (with induced distribution of strategies μ) cannot involve strategies that are **dominated** for some player $j \in \mathcal{I}$.*

A fundamental property of our equilibrium concept is that the strategy profiles that it qualifies as reasonable outcomes of a non-atomic game with a continuum of players, are also in some sense reasonable outcomes of large finite games. This property is established with Proposition 2, Proposition 3, and their corollaries, and they provide two convenient ways of checking whether or not a given strategy profile is a limit-plausible equilibrium. Proposition 2 will prove useful in *qualifying* a strategy profile as being the unique limit-plausible equilibrium of a non-atomic game with a continuum of players G, and Proposition 3 will prove useful in *disqualifying* a strategy profile from being limit-plausible equilibrium of G.

1.7 Examples - Continued

In this section we continue studying the examples of Section 1.5 including the aforementioned voting game with seats. In each example, we will use the propositions and corollaries of the previous section to justify our prediction of how the game will be played. This way we make clear the extent that our results facilitate the analysis of games with a continuum of players.

1.7.1 Voting game with seats

In this voting game there is a continuum of voters and a finite number of parties which are allocated a finite number of parliamentary seats depending on the number of votes that they receive. We assume that the more seats a party occupies the better it can serve the interests of those who voted for it.

We have shown that in this non-atomic game with a continuum of voters any voting pattern is a Nash equilibrium. The paradox is that while any single voter is not indifferent with respect to the *outcome* of the elections, she nevertheless is indifferent between any two of her own strategies, and this holds *independently* of what the other voters do. The unique equilibrium that we singled out as plausible is the one where every player votes for her most-preferable party. In light of the results of the previous section, we can now further substantiate these statements.

Consider an allegedly implausible equilibrium of this game. In such an equilibrium there exists at least one player in $[0, 1]$ say player j, who votes for a party that is *not* her most-preferred one. This strategy is not a best-response in the restricted game G_N because "voting for the most-preferred party" always results in a higher payoff for player $j \in N$.[14] Therefore, this strategy is dominated in the restricted game G_N and in any restricted game. According to Proposition 3 any strategy that involves not voting for that player's most-preferred party cannot be part of any limit-plausible equilibrium

[14]Remember that for a party more votes means at least as many seats and more seats means more effective representation of the interests of the voters for that party.

of G.

On the other hand, for player $j \in N$ the strategy "voting for the most-preferable party" is a best-response in any restricted game G_N and, as such, a μ-dominant strategy. According to Proposition 2 in a limit-plausible equilibrium of G every player votes for her most-preferable party.

1.7.2 Intertemporal welfare maximization

In this example we consider a two-period economy G with a single homogeneous good and a continuum of agents. In period 1 each agent decides how much of her endowment to save for period 2, and consumes the remainder. In period 2 due to the decisions of a benevolent policy-maker, agents make no decision and each of them consumes the average of the total savings of period 1. Previously we demonstrated that in this economy implausible equilibria arise because agents can take their utility in period 2 as given, and seek to maximize only their utility for period 1. It is not that agents do not care about their utility in period 2, but that there is no way for any single agent to affect it.

We have shown that in any restricted economy G_N the expected utility of any agent $j \in N$ is

$$u^j(c_1^j, c_2^j) := u^j(c_1^j) + u^j(\frac{1}{\#N} \sum_{i \in N} (e^i - c_1^i))$$

For any finite N individual savings can affect $\frac{1}{\#N} \sum_{i \in N} (e^i - c_1^i)$ and so, any strategy that maximizes the two-period u^j will involve strictly positive level of savings. Thus, the strategy "save nothing" is dominated by the strategy "save something". This is because we assume that no player can consume all her endowment in period 1 without becoming satiated. Consequently the agents face the dilemma of saving for period 2 or throwing away units of good. This dominance holds for any finite number of players and for any distribution of strategies of the other players. According to Proposition 3, strategy profiles where even one player saves nothing for period 2 can not be part of a limit-plausible equilibrium. Therefore in any limit-plausible equilibrium of G every player will save a strictly positive amount.

1.7.3 Cournot market game

This example involves a Cournot market game $G = \langle(\mathcal{I}, \mathbb{B}, \lambda), A, p\rangle$ with set of firms $\mathcal{I} := [0, 1]$ where each firm $j \in \mathcal{I}$ decides its level of production of a single homogeneous good $q^j \in A^j := [0, q_{max} \geq 1 - c] \cap \mathbb{Q}$ with $c \in [0, 1)$ unit cost of production and seeks to maximize its profit function

$$p^j(q^j, Q) := (1 - Q)q^j - cq^j = (1 - \int_{\mathcal{I}} q^i d\lambda - c)q^j$$

Given any finite set of firms N we have shown that firm j has an incentive to produce $q^{j^*} := \frac{1-c}{N+1}$. This incentive exists for any production level of the other firms in the market. Therefore for firm j any strategy $q^j \in [0, q_{max}] \cap \mathbb{Q} \setminus \{q^{j^*}\}$ is dominated by the strategy q^{j^*}. According to Proposition 3 any strategy $q^j \in [0, q_{max}] \cap \mathbb{Q} \setminus \{q^{j^*}\}$ cannot be part of a limit-plausible equilibrium of G. The unique symmetric Nash equilibrium of the restricted game G_N that we singled out is $\{q^j = q^{j^*} \text{ for every } j \in N\}$ and the strategy of producing $q^{j^*} := \frac{1-c}{N+1}$ remains a best-response, irrespective of the production level of the other firms. According to Corollary 1 the limit-plausible equilibrium of G that consists of the strategies q^{j^*}.

1.8 Concluding Remarks

We have identified a weakness in Nash equilibrium in a large class of games with a continuum of players. This weakness is of both conceptual and technical importance and raises issues that have not been adequately addressed in the literature.

The conceptual aspect of the problem involves the suggestion that in the non-atomic game with a continuum of players, players may choose strategies that they would not choose in *any* finite version of the game. This calls into question the relationship between games with a continuum of players and games with finitely many players, and whether games with a continuum of players are a suitable limit-abstraction of large finite games.

Through a number of examples we demonstrated two important side-effects of assuming a continuum of players. First, assuming a continuum of players breaks the link between strategies and outcomes. Since no single player can affect the outcome of the game, players become indifferent among their strategies while they are not indifferent among outcomes of the game.

The second side effect is that any strategic interaction among players is completely muted, and the strategy of any single player has no effect on any other player. Note that by construction the payoff of each player depends predominantly on the distribution of strategies of all players, and permuting the strategies of two players does not affect the payoff of the other players. Nonetheless, the fact that a specific strategy has been played should impact the payoff of the other players even if this impact is independent of the identity of the player who chose the strategy. Zero strategic interaction is a particularly striking side-effect since strategic interaction is the essence of game theory and cornerstone of Nash equilibrium.

The technical aspect of the problem involves the proliferation of Nash equilibria. In most cases infinitely many strategy profiles are qualified as Nash equilibria, and in many cases any strategy profile is a Nash equilibrium! This way the solution concept of Nash equilibrium fails its purpose and does not help predict the outcome of the game.

The weakness that we uncovered and the transpiring problems that we established have two main parts: a game and a solution concept. In this chapter we focused on the solution concept. We proposed as a solution concept of aggregative non-atomic games the "limit-plausible equilibrium", a refinement of Nash equilibrium. To construct our equilibrium concept, first we had to define a new preference relation over strategies. We said that, in the non-atomic game with a continuum of players, it is reasonable to expect players to choose strategies that they would consistently choose in large but finite versions of the game. We demonstrated in a number of strategic situations how our our equilibrium concept rules-out implausible strategy profiles but keeps the Nash equilibria that could exist in large finite-player versions of the game.

There is no consensus in the literature on whether a non-atomic game with a continuum of players is the proper way to conceptualize strategic situations with infinitely many players. Nevertheless, there is no definite alternative to the existing Schmeidler - Mas-Colell framework. In Chapter 2 of this study we take on the challenge of setting sound foundations for a new approach to formulating strategic models with infinitely many participants. This involves a new definition of the space of players and a new definition of the corresponding games. In addition, we develop a series of tools that we hope will serve as a launching platform and guide for a more consistent framework for the study of games with many players.

1.9 Appendix

1.9.1 Proof of Proposition 1

In a purely-aggregative non-atomic game with a continuum of players G every strategy profile is a Nash equilibrium.

Proof. By definition, in a purely-aggregative game G the payoff of each player $i \in \mathcal{I}$ can be written as $p^i : \Delta(A) \to \mathbb{R}$ which means that it depends only on the distribution of strategies of all players. In addition, G is a non-atomic game with a continuum of players endowed with a non-atomic measure and consequently a single player cannot affect the distribution of strategies of all players.[15] Combining the above two statements we conclude that the action of any single player does not affect her own payoff. Consequently, for a single arbitrary player, every strategy, given the distribution of strategies of all other players, leads to no greater or smaller payoff than any other strategy. Hence every strategy is a best-response for that player. Since the above player is arbitrary, the result holds for all players in the game. We see that any strategy profile of G consists of strategies that are best-responses, and as such, any strategy profile of G is also a Nash equilibrium of G. $\square$

[15]For completeness, at this point we revisit the short proof that single players have zero weight in the determination of the value of a non-atomic measure. A measure λ on $\mathcal{B}$ is non-atomic $\Leftrightarrow$ for every $B \in \mathcal{B}$ such that $\lambda(B) > 0$ there exists $K \subset B$ such that $\lambda(B) > \lambda(K) > 0$. Since the only subset of a singleton set $\{i\}$ is the empty set and $\lambda(\emptyset) = 0$, it should be the case that $\lambda(\{i\}) = 0$ for every $i \in \mathcal{B}$.

1.9.2 Proof of Proposition 2

Let G be a purely-aggregative non-atomic game with a continuum of players, such that in an arbitrarily large restricted game G_N, for player $j \in N$ there exists a μ-dominant strategy α^j. Then in a limit-plausible equilibrium of G (with induced distribution of strategies μ) player j will choose strategy α^j (or an equivalent strategy).

Proof. Assume that $G := \langle(\mathcal{I}, \mathbb{B}, \lambda), A, p\rangle$ is as described above and in some restricted game G_N, for player $j \in N$ there exists a μ-dominant strategy α^j. This means that given the distribution of strategies $\mu_{N\setminus\{j\}}$ with $j \in N$ and also given any sequence $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ that converges to μ we have

$$p^j(\alpha^j, \mu_{N\setminus\{j\}}) \geq p^j(\beta^j, \mu_N) \;\; \forall\beta^j \in A \;\text{ and so }\; \alpha^j \succsim^j_{N,\mu_{N\setminus\{j\}}} \beta^j \;\; \forall\beta^j \in A$$

Therefore, there exists a finite $m \geq \#N$ such that given any sequence of distributions of strategies $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ that converges to μ we have $\alpha^j \succsim^j_{N,\mu_{N\setminus\{j\}}}$ β^j and this holds for every $\beta^j \in A$. This means that strategy α^j is a most-preferred strategy for player j and, by definition of a limit-plausible equilibrium, player j will choose her μ-dominant strategy α^j or an equivalent strategy. □

1.9.3 Proof of Proposition 3

A limit-plausible equilibrium of a purely-aggregative non-atomic game with a continuum of players G (with induced distribution of strategies μ) cannot involve strategies that are α^j_μ-dominated for some player $j \in \mathcal{I}$.

Proof. Assume that β^j is a α^j_μ-dominated strategy for some player $j \in N$. This means that given $\mu_{N\setminus\{j\}}$ with $j \in N$ and also given any sequence $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ that converges to μ we have

$$p^j(\alpha^j, \mu_{N\setminus\{j\}}) > p^j(\beta^j, \mu_{N\setminus\{j\}}) \text{ and so } \alpha^j \succ^j_{N,\mu_{N\setminus\{j\}}} \beta^j$$

Hence for any sequence $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ that converges to μ strategy α^j is preferred by player j over strategy β^j. Therefore, in the non-atomic game with a continuum of players G, player j would never prefer β^j over α^j. It follows that the α^j_μ-dominated strategy β^j cannot be a most-preferred strategy and not a part of any limit-plausible equilibrium of G.

$\square$

Chapter 2

Complete Spaces of Games

2.1 Introduction

In the first chapter of this study we identified a weakness of the Schmeidler - Mas-Colell framework and developed a potential remedy: the limit-plausible equilibrium, a new refinement of Nash equilibrium. Nevertheless, the weakness we identified seems to extend beyond the specific equilibrium concept (Nash equilibrium) and to be intrinsic to the Schmeidler - Mas-Colell framework. Our aim is to provide sound theoretical foundations and a set of tools for an alternative framework for the study of large games. We build our foundations using familiar mathematical concepts like sequences, limits, and distributions.

We study normal-form games with countably-many[1] players and anonymous interaction. Normal-form is the most convenient representation of games of complete and perfect information where players move simultaneously and only once, which is exactly our type of game.

A limiting-game with infinitely many players is a limit abstraction that

[1]Our definition of a "countable set" includes also every finite set.

should facilitate the study of strategic situations with *finitely*-many participants since we live in a world with finitely many inhabitants. In addition, there is a discontinuity in moving from countably-many to uncountably-many players and this can give rise to problems some of which we studied in Chapter 1 where we considered a continuum of players. We believe that countability of the participants of most strategic situations is to some degree a natural property and a reasonable assumption. Therefore, we study games that involve countably-many players, as opposed to uncountably-many.

The interaction among players is anonymous in the sense that the payoff of each player depends on her action and on the distribution of strategies of all players, and permuting the strategies of two players does not affect the payoff of the other players.

Our methodology is straightforward: first we provide a new definition of a game; a definition that allows us to compare games with different number of players. Then we define a way to measure the distance between two games and this, in turn, allows us to define convergence of a sequence of games. Then we characterize a complete family of games with the following property: every game that involves all potential players is the limit of a sequence of games as the number of players increases, and every expanding sequence of games has a unique limit-game that involves all potential players. This property fosters an equivalence relation between equilibrium behavior in games with infinitely-many players and equilibrium behavior in games with finitely-many players.

In particular, in Section 2 we define a player-set in distributional form which consists of a probability measure on the space of players. This probability measure singles-out from the entire player-space, the players that are included in the specific player-set. It does so by assigning non-zero weight to the players in the game and zero weight to other players. We define sequences of player-sets in Section 3, and in Section 4 we construct a metric space of player-sets in distributional form, and show that this metric space is complete. We discuss in detail the importance of this result.

Next, in Section 5 we define a game in distributional form which is a game with a player-set in distributional form. We distinguish between an "unrestricted game" and a "restricted game". In an unrestricted game all potential players participate and it involves the entire player-space. On the contrary, in a restricted game participation is restricted to some players and it involves a subset of the player-space. We explain how, given a player-set in distributional form, an unrestricted game generates a restricted game.

In Section 6 we define sequences of games and also discuss equivalence classes of games in distributional form. In Section 7 we define the "better-response preference relation of a player" which is the ranking of strategies of a player given the strategies of the other players. This allows us to define a notion of distance between games and convergence of a sequence of games. Then we define a "limiting-game" which is an unrestricted game that is the limit of a sequence of restricted games as the number of players increases. In Section 8 we define a metric over all restricted games generated by the same limiting-game. We show that the metric space of all games generated by the same limiting-game is complete. In Section 9 we define the "best-response correspondence" of a game and establish that the related metric space of games is complete (Proposition 6). Then, in Section 10 we discuss important properties of a complete space of games regarding strategic dominance and Nash equilibrium. Finally, Section 11 concludes the chapter and the appendix includes the proofs to our propositions.

It is worth noting that we built our theoretical foundations virtually from scratch without depending on existing economics literature. The very reason that we start from scratch is that the Schmeidler - Mas-Colell framework is the only well-received approach (for the study of games with infinitely many players) in the economics and game theory literature. There are three other approaches that are mathematically elegant but have been used, at best, only by a handful of economists. We believe that one reason for this is that these approaches require that the reader has a working knowledge of either non-standard analysis, advanced set theory, or stochastic control. Khan and Sun (1999) use hyperfinite Loeb spaces, Al-Najjar (2008) uses finitely-additive measures and ultrafilters, and Lasry, Lions, and Guant (2011) use stochastic partial differential equations. Our basic tools are mathematical concepts such as sequences, limits, metric spaces, and distributions. Therefore, readers with basic knowledge of topology and measure theory should be able to follow the construction of our tools, and assess their usefulness in studying large games.

2.2 The players' space

Let $\mathcal{I}$ denote the space of potential players, a countably-infinite, closed, and metrizable set. In our case $\mathcal{I}$ is a compact metrizable space by the Heine-Borel Theorem (Aliprantis and Border (2005)) since it is closed, bounded, and metrizable. Let $\Delta(\mathcal{I})$ denote the set of all Borel probability measures on $\mathcal{I}$ endowed with the weak-convergence topology, and $d_{\Delta(\mathcal{I})}$ denote the associated Prohorov metric.[2] Then $\Delta(\mathcal{I})$ is a compact space like $\mathcal{I}$ (Theorem 15.11 Aliprantis and Border (2005)).

For each set $N \subset \mathcal{I}$ we define a Borel probability measure $\lambda_N \in \Delta(\mathcal{I})$ that satisfies the following:

1. the support[3] of λ_N consists of $\#N$ atoms (the players),

2. $\lambda_N(\{i\}) \in (0, 1]$ for each $i \in N$,

3. $\lambda_N(\{i\} \notin N) = 0$ and $\lambda_N(N) = 1$.

Note that while in our examples we consider *equiprobable* probability measures, our analytical approach applies equally to non-equiprobable measures. The support of an equiprobable probability measure λ_N consists of $\#N$ equiprobable atoms and, in addition to the above, λ_N must satisfy:

$$\lambda_N(\{i\}) \in \{0, \frac{1}{\#N}\} \text{ for each } i \in \mathcal{I}$$

This means that if i is one of the $\#N$ equiprobable atoms of N then

$$\lambda_N(\{i\}) = \frac{1}{\#N} \text{ and 0 otherwise.}$$

In the following definition we make precise the notion of a player-set in distributional form. Typically, a player-set contains the labels of the players that participate in a game. Two different player-sets may involve not only

[2]The Prohorov metric on a space $\mathcal{I}$ with σ-algebra $\mathcal{B}(\mathcal{I})$ is

$$\rho(\mu, \nu) := \inf_{\epsilon}\{\epsilon > 0 \mid \forall E \in \mathcal{B}(\mathcal{I}) | \nu(E) \leq \mu(N_\epsilon(E)) + \epsilon \text{ and } \mu(E) \leq \nu(N_\epsilon(E)) + \epsilon\}$$

where $N_\epsilon(E)$ denotes the ball of radius ϵ around E, and $\mu, \nu \in \Delta(\mathcal{I})$.

[3]Given any measure , the *support* of λ is defined as the smallest closed set $S \subset \mathcal{I}$ such that $\lambda(S) = 1$. Since $\mathcal{I}$ is a compact metrizable space the support of λ always exists (Chapter II Parthasarathy (1967) and Theorem 12.14 Aliprantis and Border (2005)).

different players but also different number of players. The distributional form maps each player-set to a probability measure whose support consists of that specific player-set. This way the distributional form provides a convenient and straightforward way of comparing player-sets of different size and contents.

Definition 14. *The* ***distributional form*** *of a player-set $N \subset \mathcal{I}$ consists of a probability measure $\lambda_N \in \Delta(\mathcal{I})$ whose support is the player-set N. The* ***weight*** *of player $i \in \mathcal{I}$ is a positive number less than 1 if $i \in N$ and 0 if $i \notin N$.*

2.2.1 Example: Sets of firms

Consider the following set of firms in a market game

$$\mathcal{I} := \{1, \frac{1}{2}, ..., \frac{m}{n}, ... \mid m < n \text{ and } m, n \in \mathbb{N}\} \cup \{0\}$$

Then the distributional form of the finite set of firms

$$B := \{1, \frac{1}{2}, ..., \frac{9}{10}\} \subset \mathcal{I}$$

is λ_B such that:

$$\lambda_B(\{1\}) = \lambda_B(\{\frac{9}{10}\}) = \lambda_B(\{\frac{m}{n} \mid 0 < m < n \leq 10 \text{ and } m, n \in \mathbb{N}\}) = \frac{1}{34}$$

and 0 otherwise.

Consider a different finite set of firms

$$K := \{1, \frac{1}{4}, \frac{1}{9}, ..., \frac{m^m}{n^n}, ... \frac{9^9}{10^{10}} \mid 0 < m < n \leq 10 \text{ and } m, n \in \mathbb{N}\} \subset \mathcal{I}$$

The distributional form of K is λ_K) such that

$$\lambda_K(\{1\}) = \lambda_K(\{\frac{m^m}{n^n} \mid 0 < m < n \leq 10 \text{ and } m, n \in \mathbb{N}\}) = \frac{1}{33}$$

and 0 otherwise. Note that $B \not\subset K$ since $\frac{1}{2}$ belongs to B but not to K. Also $K \not\subset B$, $\frac{9^9}{10^{10}}$ belongs to K but not to B.

We can proceed to compare the two sets and see the advantages of the distributional form. The original sets B and K have not equal cardinality, and also have different contents i.e. different players. On the contrary, λ_B and λ_K are probability measures defined on $\mathcal{I}$ and, as a result, comparison is possible.

2.3 Sequences of player-sets

Our approach is based on limits as the number of players increases, and therefore it is necessary before we define sequences of games, that we first define sequences of player-sets. In this section we define the conditions that guarantee that certain sequences of player-sets, we call them "admissible sequences", simulate more accurately the strategic environments that we want to study. These sequences are "expanding" and "exhaustive"; they add players to the game without excluding any of the current players, and eventually include all potential players. These are important properties since when we define a limit-game, we want it to be the limit of admissible sequences and to involve all players, a necessary requirement for uniqueness of that limit-game.

The rationale behind the admissibility requirement on sequences of games is the following:

(a) **Expanding:** We study the strategic behavior of players as the number of participants in the game increases. Therefore we are *adding* players to the existing players of a game.

(b) **Exhausting:** We want to add players to an existing strategic situation *without* systematically excluding players. We want eventually all players to participate in the game.

Definition 15. *A sequence of player-sets $\{N_i\}_{i=1}^{\infty}$ is **admissible** and we can write $N_i \uparrow \mathcal{I}$, if it is expanding and eventually involves all potential players. Thus, we require that $\{N_i\}_{i=1}^{\infty}$ converges to $\mathcal{I}$ from below i.e.*

$$N_i \subset N_{i+1} \;\; \textit{for every } i, \textit{ and } \; \bigcup_{i=1}^{\infty} N_i = \mathcal{I}$$

2.3.1 Example: Admissible sequences of firm-sets

An example of an expanding sequence of firm-sets is

$$\{B_i\}_{i<j=2}^{\infty} \quad \text{where} \quad B_i := \{1, \frac{1}{2}, ..., \frac{i}{j} \mid i < j \;\; \text{and} \;\; i, j \in \mathbb{N}\}$$

Another example of an expanding sequence of firm-sets is

$$\{Z_k\}_{k=2}^{\infty} \;\; \text{where} \quad Z_k := \bigcup_{i<j=1,...,k} W_j$$

and

$$W_j := \{1, \frac{1}{4}, ..., \frac{i^i}{j^j} \mid i < j \;\; \text{and} \;\; i, j \in \mathbb{N}\}$$

While both sequences are expanding only $\{B_i\}$ is admissible. $B_i \uparrow \mathcal{I}$, but since for example $\frac{1}{2} \notin Z_\infty$, Z_k does not converge to $\mathcal{I}$ and is not admissible.

2.3.2 Admissible sequences of player-sets in distributional form

Definition 16. *A sequence of player-sets in distributional form* $\{(\lambda_{N_i})\}_{i=1}^{\infty}$ *is* ***admissible*** *if it is generated by an admissible sequence of player-sets* $\{N_i\}_{i=1}^{\infty}$.

In the coming sections we construct various metric spaces and we consider exclusively admissible sequences.

2.4 A metric space of player-sets in distributional form

In this section we define the distance between two player-sets in distributional form and construct a complete metric space of player-sets. In this complete metric space every player-set of infinitely-many players is the limit of an expanding sequence of countable player-sets. Equivalently, every expanding sequence of player-sets has a unique limit-set, up to an equivalence class, that involves all potential players.

Proposition 4. *The metric space* $(\Delta(\mathcal{I}), d_{\Delta(\mathcal{I})})$ *of player-sets in distributional form is a complete metric space.*

A proof is provided in the Appendix.

2.4.1 Example: A metric space of firms

As previously $d_{\Delta(\mathcal{I})}$ denotes the Prohorov metric on $\Delta(\mathcal{I})$.

Consider the following two player-sets in distributional form λ_B such that

$$\lambda_B(\{1\}) = \lambda_B(\{\frac{1}{2}\}) = \lambda_B(\{\frac{m}{n} \mid 0 < m < n \leq 10 \text{ and } m, n \in \mathbb{N}\}) = \frac{1}{34}$$

and 0 otherwise, and λ_K such that

$$\lambda_K(\{\frac{m^m}{n^n} \mid 0 < m < n \leq 10 \text{ and } m, n \in \mathbb{N}\}) = \frac{1}{33} \quad \text{and 0 otherwise.}$$

Then the distance between these two player-sets in distributional form is equal to the corresponding value of the Prohorov metric.

We understand that calculating the value of the Prohorov metric can be a challenging task. As we will explain later, a researcher does not have to deal with this numerical calculation in order to use our tools and results.

2.5 Games in distributional form

Let (A, d_A) denote a non-empty compact metric space of actions, $\Delta(A)$ denote the set of Borel probability measures on A equipped with the topology of weak convergence, and $d_{\Delta(A)}$ denote the Prohorov metric on $\Delta(A)$. Then $(\Delta(A), d_{\Delta(A)})$ is a compact metric space (Theorem 6.4, Chapter II Parthasarathy (1967)).

Definition 17. *A non-cooperative* ***game in distributional form*** *is a triple*

$$G_N := \langle \lambda_N, A, p_N \rangle$$

consisting of a player-set λ_N in distributional form, an action space A, and a *family of continuous payoff functions*

$$p^i_N : A \times \Delta(A) \to \mathbb{R} \ \ \textit{for each} \ \ i \in N$$

2.5.1 Example: Cournot market game in distributional form

Consider the following Cournot market game $G_B = \langle \lambda_B, A, p_B \rangle$ with

$$\lambda_B(\{\frac{m}{n} \mid 0 < m < n \leq 10 \ \text{ and } \ m, n \in \mathbb{N}\}) = \frac{1}{34} \ \text{ and 0 otherwise.}$$

Each firm $j \in B$ chooses a level of production of a single homogenous good

$$q^j \in A^j \subseteq A := [0, q_{max}] \cap \mathbb{Q}$$

and faces $c \in [0, 1)$ unit cost of production. Assume that $P(Q) = 1 - Q$ relates the price P of the good with the average quantity available $Q := \frac{1}{34} \sum_{i \in B} q^i$. Then the payoff function of firm j is its profit function

$$p^j(q^j, \frac{1}{34} \sum_{i \in B} q^i; c) := (1 - \frac{1}{34} \sum_{i \in B} q^i - c)q^j$$

We see that all parameters of the game: player-set, action space, and payoff functions, are defined and consequently the Cournot market game G_B is well-defined.

2.5.2 Unrestricted games and games of restricted participation in distributional form

Definition 18. *A non-cooperative* ***unrestricted game*** *in distributional form is a triple*

$$G_{\mathcal{I}} := \langle (\lambda), A, p \rangle$$

where $\lambda \in \Delta(\mathcal{I})$ is a probability measure on $\mathcal{I}$, A is a space of actions, and p is a family of continuous payoff functions

$$p^i : A \times \Delta(A) \to \mathbb{R} \quad \text{for each} \quad i \in \mathcal{I}$$

Definition 19. *Given an unrestricted game $G_{\mathcal{I}}$ in distributional form, the* ***restricted game*** *in distributional form with player-set $N \subset \mathcal{I}$ is denoted by*

$$G_N := \langle \lambda_N, A, p_N \rangle$$

consisting of a player-set in distributional form λ_N, an action space A, and a family p_N of continuous payoff functions

$$p^i_N : A \times \Delta(A) \to \mathbb{R} \quad \text{for each} \quad i \in N$$

being the restriction of the original family of payoffs functions p to the player-set N.

The purpose of the restricted game is to simulate the strategic environment that would arise if participation in the game was restricted to a subset of players.

2.5.3 Example: Restricted Cournot market games

An example of an unrestricted game Cournot market game is $G_{\mathcal{I}} = \langle \lambda, A, p \rangle$ where the action space A is the same as in the previous example, and the payoff function of a firm $j \in \mathcal{I}$ is

$$p^j(q^j, \int_{\mathcal{I}} q^i d\lambda; c) := (1 - \int_{\mathcal{I}} q^i d\lambda - c)q^j$$

A *restricted* Cournot market game is

$$G_B = \langle \lambda_B, A, p_B \rangle$$

with

$$\lambda_B(\{\frac{m}{n} \mid 0 < m < n \leq 10 \ \text{ and } \ m, n \in \mathbb{N}\}) = \frac{1}{34}$$

and 0 otherwise. The action space is the same, and the payoff function of a firm $j \in B$ is

$$p_B^j(q^j, \frac{1}{34} \sum_{i \in B} q^i; c) := (1 - \frac{1}{34} \sum_{i \in B} q^i - c)q^j$$

Another restricted game of the Cournot market game is

$$G_K = \langle \lambda_K, A, p_K \rangle$$

with

$$\lambda_K(\{\frac{m^m}{n^n} \mid 0 < m < n \leq 10 \ \text{ and } \ m, n \in \mathbb{N}\}) = \frac{1}{33}$$

and 0 otherwise. The action space is the same, and the payoff function of a firm $j \in K$ is

$$p_K^j(q^j, \frac{1}{33}\sum_{i\in K} q^i; c) := (1 - \frac{1}{33}\sum_{i\in K} q^i - c)q^j$$

2.6 Sequences of games

Consider the family of all sequences $\{N_i\}_{i=1}^{\infty}$ of expanding player-sets $N_i \subseteq N_{i+1} \subset \mathcal{I}$. Such a sequence of player-sets induces a sequence of games $\{G_{N_i}\}_{i=1}^{\infty}$ with increasing number of players and where all players of G_{N_i} participate also in every game $G_{N_{i+j}}$ further in the sequence. We will consider sequences of games that in addition to the above, at the limit exhaust the player-space and involve all players.

Definition 20. *A sequence of games with countably many players $\{G_N\}_{\#N\geq 2}$ is* ***admissible****, and we can write $G_N \uparrow G_{\mathcal{I}}$, if it is generated by an admissible sequence of player-sets $\{N_i\}_{i=1}^{\infty}$.*

2.6.1 Example: Sequences of Cournot market games

In our example of Cournot market game the set of firms is

$$\mathcal{I} := \{1, \frac{1}{2}, ..., \frac{m}{n}, ... \mid m < n \ \text{ and } \ m, n \in \mathbb{N}\} \cup \{0\}$$

In the example in Section 2.3.1 of player-sets the sequence $\{B_i\}$ is expanding and converges to $\mathcal{I}$, while $\{Z_k\}$ may be expanding but does not converge to $\mathcal{I}$. Thus the sequence of games induced by $\{B_i\}$ is admissible and is denoted by $\{G_{B_i}\}_{i=1}^{\infty}$ while $\{Z_k\}$ is non-admissible and on that account also the corresponding sequence of games $\{G_{Z_k}\}$ is not admissible.

2.6.2 Equivalence of games

In order to construct a complete metric space of games where every convergent sequence of games has a unique limit (which is an unrestricted game) up to an equivalence class, we first need to define when two games are considered to be equivalent. In our case an equivalence class is the set of all game-sequences that are considered to be equivalent because they converge to the same unrestricted game.

Our notion of equivalence is based on the observation that in a game strictly increasing affine transformations of the payoffs do not change a player's preference ranking over strategies. This is related to the fact that cardinal utility functions are unique only up to strictly increasing affine transformations.

Definition 21. *Two payoff functions*

$$p^i : A \times \Delta(A) \to \mathbb{R}$$

and

$$\tilde{p}^i : \tilde{A} \times \Delta(\tilde{A}) \to \mathbb{R}$$

are ***equivalent*** *and write* $p^i \sim \tilde{p}^i$ *if there exist scalars* $\theta^i > 0$ *and* ϕ^i *such that*

$$p^i = \theta^i \tilde{p}^i + \phi^i$$

Definition 22. *Two games* $G_N := \langle \lambda_N, A, p_N \rangle$ *and* $\tilde{G_N} := \langle \tilde{\lambda}_N, \tilde{A}, \tilde{p}_N \rangle$ *are* ***equivalent*** *and we write* $G_N \sim \tilde{G}_N$ *if for every* $i \subset N$ *there exist scalars* θ^i *and* ϕ^i *such that* $p^i \sim \tilde{p}^i$.

2.6.3 Example: Equivalent Cournot market games

Consider our aforementioned unrestricted game Cournot market game $G_{\mathcal{I}} = \langle \lambda, A, p \rangle$ where the function $P(Q) = 1 - Q$ relates the price P of a single homogeneous good with the average quantity available $Q := \int_{\mathcal{I}} q^i d\lambda$, and the per unit cost is $c > 0$ for $i \in \mathcal{I}$. In addition, consider also the unrestricted game Cournot market game $\tilde{G} = \langle \lambda, A, \tilde{p} \rangle$ where the relation of the price $\tilde{P}$ of the good with the average quantity available is determined by $\tilde{P}(Q) = \theta(1 - Q)$, and the per unit cost is now $\theta c > 0$ for $i \in \mathcal{I}$ with θ a positive real number.

In any restricted game G_N the payoff function is of the form

$$p_N^j(q^j, \frac{1}{\#N}\sum_{i \in N} q^i; c) := (1 - \frac{1}{\#N}\sum_{i \in N} q^i - c)q^j$$

On the other hand, in any restricted game $\tilde{G}^N$ the payoff function is of the form

$$\tilde{p}_N^j(q^j, \frac{1}{\#N}\sum_{i \in N} q^i; \theta c) := \theta(1 - \frac{1}{\#N}\sum_{i \in N} q^i - c)q^j = \theta p_N^j$$

and therefore $p_N^j \sim \tilde{p}_N^j$. Since this holds for every $j \in \mathcal{I}$, we have $G \sim \tilde{G}$ and the two unrestricted games are equivalent.

In the following sections we consider exclusively player-sets and games in distributional form and we will refer to them simply as "player-sets" and "games"; we will restore the full title if ambiguity is likely.

2.7 Better-Response preference relation and correspondence

2.7.1 The graph of the Better-Response preference relation of a player

In this section we define the "better-response preference relation" of a player. This preference relation is basically a player's ranking of her strategies, given a distribution of strategies of all other players. The better-response preference relation is a set whose elements are the outcomes of the following process: compare any two strategies of a player given the distribution of strategies of all other players, and choose the strategy that provides a higher payoff to that player. If two strategies provide equal payoff given the specific distribution of strategies of all other players, then they are considered to be "equivalent". The better-response preference relation is closely related to the Nash equilibrium concept since it is a preference relation based on a comparison of payoffs given what the other players have chosen to do.

Note that a better-response strategy is not necessarily a dominant strategy. This happens for two reasons:

(1) a better-response strategy provides to a player at least as high payoff as the *specific* strategy that is compared to, and not higher payoff than *any* other strategy

(2) a better-response strategy provides to a player at least as high payoff as a strategy that is compared to, given a *specific* distribution of strategies of all other players, and not for *any* distribution of strategies of all other

players

On the contrary, a dominant strategy is necessarily a better-response strategy since it provides to a player higher payoff than any other strategy, given *any* distribution of strategies of the other players.

In general, equilibrium behavior can be captured by considering the "best-response preference relation" which qualifies the strategies that provide to the player at least as high payoff as *any* other strategy, given the distribution of strategies of all other players. This way however, we lose information since we cannot know how this player ranks the strategies that are not best-responses. From the better-response preference relation it is always possible to deduce the best-response preference relation by identifying the strategies that are preferred over *any* other strategy (or that are equivalent). On the contrary, in most cases it is not possible to recover the better-response preference relation just by examining the best-response preference relation.

Definition 23. *Given a game G_N, the* ***better-response preference relation*** *over strategies of player $j \in N$ is a family of binary relations*

$$\succsim^j \quad \text{on } \; A \times A \text{ and indexed by } \; \Delta_{\#N-1}(A)$$

that satisfies the following requirements:

(1) $\alpha \succ^j_{N,\mu_{N\setminus\{j\}}} \beta$ *if* $p^j(\alpha, \mu_{N\setminus\{j\}}) > p^j(\beta, \mu_{N\setminus\{j\}})$ *and this means that* $\alpha \succsim^j_{N,\mu_{N\setminus\{j\}}} \beta$ *but not* $\beta \succsim^j_{N,\mu_{N\setminus\{j\}}} \alpha$

(2) $\alpha \sim^j_{N,\mu_{N\setminus\{j\}}} \beta$ *if* $p^j(\alpha, \mu_{N\setminus\{j\}}) = p^j(\beta, \mu_{N\setminus\{j\}})$ *and this means that* $\alpha \succsim^j_{N,\mu_{N\setminus\{j\}}} \beta$ *and also* $\beta \succsim^j_{N,\mu_{N\setminus\{j\}}} \alpha$

*The first requirement states that in G_N, player j will **prefer** strategy α over strategy β if strategy α provides her with a higher payoff than strategy β given the distribution $\mu_{N\setminus\{j\}}$ of strategies of all other players.*

*The second requirement states that in G_N, player j will be **indifferent** between strategy α and strategy β if strategy α provides her with the same payoff as strategy β given the distribution $\mu_{N\setminus\{j\}}$ of strategies of all other players.*

Recall that the graph of a function $f : X \to Y$ is the set $\{(x, y)|\ y = f(x)\}$ and contains both the argument and the image. This way the graph of a preference relation $f : A \times A \times \Delta(A) \to A$ can be expressed as the set $\{(\alpha, \beta, \mu, \alpha)|\ \alpha \succ \beta\}$. This means that when we compare strategy α with strategy β, given the distribution μ, the outcome of the comparison is α (that provides higher payoff).

Since we are comparing two strategies and we have to choose one of them, the range is constrained to one of the two strategies under consideration. Hence we could simplify notation and write $\{(\alpha, \beta, \mu)|\ \alpha \succ \beta\}$. In the case where α and β are equivalent and provide the same level of payoff, the graph of this preference relation will contain both (α, β, μ) and (β, α, μ).

Most importantly, the graph of a preference relation contains the complete structure of that preference relation, and knowing the graph we can deduce the player's preferences over her strategies.

2.7.2 Example: A game of coordination

Consider a game of coordination where a finite number of players is trying to coordinate and choose the same strategy. In this game a player receives maximum payoff if she chooses the same strategy with the majority of the players, and minimum payoff otherwise. Assume that $A := \{pull,\ push\}$ is the set of the two available strategies. Then the graph of the better-response preference relation of a player is one of the following three:

(a)

$$\{(push,\ pull,\ \text{the majority of other players have chosen “push”})\}$$

the above graph indicates that given the distribution of strategies “the majority of other players have chosen to push” the strategy “push” provides a higher payoff than the strategy “pull”.

(b)

$$\{(pull,\ push,\ \text{the majority of other players have chosen “pull”})\}$$

the above graph indicates that given the distribution of strategies “the majority of other players have chosen to pull” the strategy “pull” provides a higher payoff than the strategy “push”.

If we have odd number of players there is an additional case where the graph of the better-response preference relation of a player consists of two elements

(c)

$$\{(push,\ pull,\ \text{half of the players have chosen “push” and half have chosen “pull”}),$$

$$(pull,\ push,\ \text{half of the players have chosen “push” and half have chosen “pull”})\}$$

indicating that given this distribution of strategies "half of the players have chosen to push and half have chosen to pull" the strategy "pull" provides the same payoff with the strategy "push".

2.7.3 Limiting-games and the graph of the Better-Response correspondence of a game

Now we define the graph of the better-response correspondence of a game which is the union of the graphs of better-response preference relations for all players in the game.

Definition 24. *Given a game G_N* ***the graph of the better-response correspondence*** *is the set*

$$\Gamma_N := \{(j, \lambda_N, \alpha, \beta, \mu_{N\setminus\{j\}}) \in \mathcal{I} \times \Delta(\mathcal{I}) \times A^2 \times \Delta(A)\}$$

with strategies α and β such that

$$p^j(\alpha, \mu_{N\setminus\{j\}}) \geq p^j(\beta, \mu_{N\setminus\{j\}})$$

A typical element of this set indicates that for a player $j \in N$ in the game with player-set λ_N, given the distribution of strategies of all other players $\mu_{N\setminus\{j\}} \in \Delta_{\#N-1}(A)$, strategy α that is listed first (compared to strategy β) provides player j with at least as high payoff as strategy β that is listed second (compared to strategy α).

Definition 25. *An unrestricted game $G_{\mathcal{I}}$ is a* ***limiting-game*** *if it is the limit of an admissible sequence of games $\{G_N\}_{\#N=2}^{\infty}$ as the number of play-*

ers increases. Equivalently, an unrestricted game $G_{\mathcal{I}}$ is a limiting-game if its better-response correspondence is the limit of a sequence of better-response correspondences of restricted games $\{G_N\}_{\#N=2}^{\infty}$.

We believe that the above requirement on better-response correspondence is sufficient for a limiting-game to be a meaningful limit-abstraction of the strategic environment of a large finite game. Under the above definition, when a sequence of games converges to a limiting-game, we know that the better-response correspondence of the limiting-game is the limit of a sequence of better-response correspondences of the respective games. This has an especially useful consequence: for any single player j in the game, given a distribution of strategies of all other players which is the limit of a sequence of restricted distributions of strategies, player j will rank her strategies in the limiting-game approximately the same way that she would in a large finite game. In turn, this has profound implications for the properties of Nash equilibrium in such a metric space of games: we can avoid discontinuities in the better-response preference relation that give rise to implausible preferences over strategies and lead to implausible equilibria.

It is worth noting that in a limiting-game that is the limit of an admissible sequence of games, a player may not have a well-defined payoff function. This is not a problem because what we need is that a player, given a distribution of strategies of all other players in the game, can compare any two of her strategies based on the payoff that they provide to her. A well-defined payoff function would additionally allow a player to compare any two of her strategies given *different* distributions of strategies of all other players; this is something that we do not require for our analysis.

2.8 A metric space of games in distributional form

In this section we define the distance between two games and then construct a metric space of games in distributional form. Our analytical approach consists of embedding the complete metric space of player-sets in distributional form into a metric space of games in distributional form. Under this approach the completeness property follows directly. This allows our results and the corresponding proofs in the appendix to be particularly straightforward. In this complete metric space of games, every game with infinitely-many players is the limit of a sequence of countable games as the number of players increases. Alternatively, every convergent sequence of games has a unique limiting-game, up to an equivalence class.

2.8.1 The distance between games

We define the distance between any two games as the Hausdorff distance between the graphs of their better-response correspondences.

Consider the following metrics: the metric $d_{\mathcal{I}}$ with respect to which $\mathcal{I}$ is a complete metric space[4], the Prohorov metric $d_{\Delta(\mathcal{I})}$ on $\Delta(\mathcal{I})$, the metric d_A on A, and the Prohorov metric $d_{\Delta(A)}$ on $\Delta(A)$.

Let

$$d_1 := \frac{d_{\mathcal{I}} + d_{\Delta(\mathcal{I})} + 2d_A + d_{\Delta(A)}}{5}$$

[4]Such a metric exists because $\mathcal{I}$ is a compact metrizable space by Theorem 3.28 in Aliprantis and Border (2005).

be a product metric on

$$\mathcal{I} \times \Delta(\mathcal{I}) \times A^2 \times \Delta(A)$$

The properties of semi-definiteness, symmetry, and triangle inequality of $d_{\mathcal{I}}$, $d_{\Delta(\mathcal{I})}$, d_A, and $d_{\Delta(A)}$ are preserved under addition and scalar multiplication and therefore d_1 is a well-defined metric on $\mathcal{I} \times d_{\Delta(\mathcal{I})} \times A^2 \times \Delta(A)$.

For sets of players $N, \tilde{N} \subset \mathcal{I}$ the distance between any two games G_N and $G_{\tilde{N}}$ is defined as follows

$$d_g(G_N, G_{\tilde{N}}) := \max\{\sup_{x \in \Gamma_N} \inf_{\tilde{x} \in \Gamma_{\tilde{N}}} d_1(x, \tilde{x}), \sup_{\tilde{x} \in \Gamma_{\tilde{N}}} \inf_{x \in \Gamma_N} d_1(x, \tilde{x})\}$$

2.8.2 Example: The distance between restricted Cournot market games

We will apply our definition of the distance between two games to the restricted Cournot market games $G_B = \langle \lambda_B, A, p_B \rangle$ and $G_K = \langle \lambda_K, A, p_K \rangle$.

The graph of G_B is

$$\Gamma_B := \{(j, \lambda_B, \alpha^j, \beta^j, \frac{1}{34} \sum_{i \in B \setminus \{j\}} q^i) \ \in \ B \times \Delta_B(\mathcal{I}) \times A^2 \times \Delta_{34}(A)\}$$

with strategies α^j and β^j such that

$$(1 - \frac{1}{34}(\sum_{i \in B \setminus \{j\}} q^i + \alpha^j) - c)\alpha^j \ \geq \ (1 - \frac{1}{34}(\sum_{i \in B \setminus \{j\}} q^i + \beta^j) - c)\beta^j$$

The graph of G_K is

$$\Gamma_K := \{(j, \lambda_K, \tilde{\alpha}^j, \tilde{\beta}^j, \frac{1}{34}\sum_{i\in K\setminus\{j\}} q^i) \ \in K \ \times \Delta_K(\mathcal{I}) \times A^2 \times \Delta_{34}(A)\}$$

with strategies $\tilde{\alpha}^j$ and $\tilde{\beta}^j$ such that

$$(1 - \frac{1}{34}(\sum_{i\in K\setminus\{j\}} q^i + \tilde{\alpha}^j) - c)\tilde{\alpha}^j \ \geq \ (1 - \frac{1}{34}(\sum_{i\in K\setminus\{j\}} q^i + \tilde{\beta}^j) - c)\tilde{\beta}^j$$

then the distance between G_B and G_K is

$$d_g(G_B, G_K) := \max\{\sup_{x\in\Gamma_B} \inf_{\tilde{x}\in\Gamma_K} d_1(x, \tilde{x}), \sup_{\tilde{x}\in\Gamma_K} \inf_{x\in\Gamma_B} d_1(x, \tilde{x})\}$$

2.8.3 Short note on the numerical calculation of the Hausdorff and Prohorov metrics

We understand that calculating the value of the Hausdorff metric or the value of the Prohorov metric can be a challenging task. Nonetheless, a researcher does not have to deal with these numerical calculations in order to use our tools and results. Analogously to the way that our proofs establish the *validity* of our results, our examples aim to strengthen the *intuition* for the underlying mechanism of our methodology. These examples are not necessarily a "user guide" for researchers. Provided that a strategic situation can be modeled as a game that satisfies our assumptions about the player-space, the action-space, and the payoff functions, a researcher can use our analytical tools and invoke our propositions to analyze this strategic situation.

2.8.4 Completeness and the Better-Response correspondence

Proposition 5. *The metric space* $(\mathcal{G},\ d_g)$ *is complete.*

A proof is provided in the Appendix.

In effect, Proposition 5 establishes the upper and lower hemicontinuity property of the better-response correspondence of the games that belong to a complete metric space $(\mathcal{G},\ d_g)$. For a game $G_L \in \mathcal{G}$ the following two properties are true:

(a) lower hemicontinuity: the graph of the better-response correspondence of G_L is the limit of a sequence of graphs of better-response correspondences of restricted games G_M with $M \subset L$

(b) upper hemicontinuity: the limit of any sequence of graphs of better-response correspondences of restricted games $G_M \in \mathcal{G}$ is the graph of the better-response correspondence of a game $G_L \in \mathcal{G}$

The better-response correspondence includes the entire structure of players' preferences over their strategies. As such, the better-response correspondence also includes the "best-response correspondence" that we will define in the next section. The best-response correspondence is just a part of the structure of players' preferences over their strategies but it plays a key role in the definition of Nash equilibrium since it characterizes a player's "optimal" strategies given what the other players have chosen to do. We have established the upper and lower hemicontinuity of the better-response correspondence of every game that belongs to $(\mathcal{G},\ d_g)$.

2.9 Nash equilibrium and the Best-Response correspondence

First, given a game G_N, for each player $i \in N$ we characterize the set of "best" strategies for a given situation. We define for each player $i \in N$ the set of all strategies that, given the strategies of all other players, provide to player i at least as high payoff as *any* other strategy. Then, a strategy profile is a Nash equilibrium of a game if every player employs a "best" strategy (see Definition 23 below).

Definition 26. *Given a game $G_N := \langle \lambda_N, A, p_N \rangle$, the* ***best-response correspondence*** *is the set-valued function $Br : A \times \Delta(A) \to 2^A$ where 2^A is the family of all subsets of A. The set of best-responses for player $i \in N$ is*

$$Br^i(\mu_{N\setminus\{i\}}) := \{\alpha^i \in A|\ p^i_N(\alpha^i; \mu_{N\setminus\{i\}}) \geq p^i_N(\beta^i; \mu_{N\setminus\{i\}}) \ \text{ for every } \ \beta^i \in A\}$$

Definition 27. *Given a game $G_N := \langle \lambda_N, A, p_N \rangle$, a strategy profile $\mathcal{A}_N$ where α^i is the strategy of player i is a* ***Nash equilibrium*** *of G_N if*

$$\alpha^i \in Br^i(\mu_{N\setminus\{i\}}) \ \text{ for every } \ i \in N$$

2.9.1 The graph of the Best-Response correspondence of a game

Definition 28. *The* ***graph of the best-response correspondence*** *of G_N is the set*

$$\{(i, \lambda_N, \alpha, \mu_{N\setminus\{i\}})\} \ \in \ \mathcal{I} \times \Delta(\mathcal{I}) \times A \times \Delta(A)\}$$

A typical element of this set indicates that:

- For player $i \in \mathcal{I}$ in the game with player-set λ_N,

- given the distribution $\mu_{N \setminus \{i\}}$ of strategies of all other players, that:

*- strategy α provides to player i at least as high payoff as **any** other strategy*

2.9.2 Completeness and the Best-Response correspondence

Proposition 6. *The metric space $(\mathcal{B}, d_2)$ of graphs of best-response correspondences is a complete metric space with respect to the product metric*

$$d_2 := \frac{d_{\mathcal{I}} + d_{\Delta(\mathcal{I})} + d_A + d_{\Delta(A)}}{4} \quad \text{on} \quad \mathcal{B} := \mathcal{I} \times \Delta(\mathcal{I}) \times A \times \Delta(A)$$

A proof is provided in the Appendix.

The above result establishes that the best-response correspondence of a limiting-game is the limit of a sequence of best-response correspondences of restricted games, and also that when a sequence of games converges to a limiting-game, the limit of the sequence of graphs of best-response correspondences is the best-response correspondence of that limiting-game.

2.10 Strategic Dominance in Restricted games and Nash equilibria of Limiting-games

The following result connects directly the first two chapters of this study. In Chapter 1 we constructed a refinement of Nash equilibrium and demonstrated how it qualifies only plausible outcomes as equilibria of a game. Below we discuss why an arbitrary game in our space of games has only plausible Nash equilibria, and therefore we should expect no discontinuities in equilibrium behavior as the number of players increases.

2.10.1 Dominant strategies

Recall from Chapter 1 that a strategy is **μ-dominant** if it provides at least as high payoff to player j as any other strategy given $\mu_{N\setminus\{j\}}$ and for any sequence $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ of distributions of strategies of all other players that converges to μ. A μ-dominant strategy is not necessarily a "dominant strategy". This is because a μ-dominant strategy provides to j at least as high payoff as any other strategy, *only* for $\mu_{N\setminus\{j\}}$ or for sequences $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ that converge to μ, but *not* for *any* distribution of strategies of all other players. On the contrary, a dominant strategy is necessarily a μ-dominant strategy since it provides to j higher payoff than any other strategy, given *any* distribution of strategies of the other players.

Note that if a limiting-game G belongs to a complete metric space of games $\mathcal{G}$, and in an arbitrarily large restricted game $G_N \in \mathcal{G}$ for player $j \in N$ there exists a **μ-dominant** strategy α^j then in a Nash equilibrium of G (with induced distribution of strategies μ) player j will choose strategy α^j (or an equivalent strategy).

The above statement is true because, in a complete metric space of games, the best-response correspondence of a limiting-game is the limit of a sequence of best-response correspondences of restricted games. Therefore if strategy α^j is a best-response given $\mu_{N\setminus\{j\}}$ and for any sequence $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$

of distributions of strategies of all other players that converges to μ, then strategy α^j also is a best-response in the limiting-game given μ and in a Nash equilibrium of G (with induced distribution of strategies μ) player j will choose strategy α^j (or an equivalent strategy).

It follows that if a limiting-game G belongs to a complete metric space of games $\mathcal{G}$, and in an arbitrarily large restricted game $G_N \in \mathcal{G}$ for player $j \in N$ there exists a **dominant** strategy α^j then in a Nash equilibrium of G (with induced distribution of strategies μ) player j will choose strategy α^j (or an equivalent strategy).

2.10.2 Dominated strategies

Recall from Chapter 1 that a strategy β^j is α^j_μ**-dominated** if it provides lower payoff to player j than the strategy α^j given $\mu_{N\setminus\{j\}}$ and for any sequence $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ of distributions of strategies of all other players that converges to μ. An α^j_μ-dominated strategy is not necessarily a "dominated strategy". This happens for two reasons: first because an α^j_μ-dominated strategy is not dominated in given *any* distribution of strategies of all other players, but only given the *given* $\mu_{N\setminus\{j\}}$ and for any sequence $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ of distributions of strategies of all other players that converges to μ. On the contrary, a dominated strategy is necessarily an α^j_μ-dominated strategy since it is dominated given *any* distribution of strategies of the other players.

Note that if a limiting-game G belongs to a complete metric space of games $\mathcal{G}$ then a Nash equilibrium of G (with induced distribution of strategies μ) cannot involve strategies that are α^j_μ**-dominated** for some player $j \in \mathcal{I}$.

The above statement is true because, in a complete metric space of games, the best-response correspondence of a limiting-game is the limit of a sequence of best-response correspondences of restricted games. Therefore if strategy β^j is *not* a best-response given $\mu_{N\setminus\{j\}}$ and for any sequence $\{\mu_{N\setminus\{j\}}\}_{\#N\geq m}$ of distributions of strategies of all other players that converges to μ, then strategy β^j also is *not* a best-response in the limiting-game given μ and in a Nash equilibrium of G (with induced distribution of strategies μ) player j will *not* choose strategy β^j.

It follows that G if a limiting-game G belongs to a complete metric space of games $\mathcal{G}$ then a Nash equilibrium of G (with induced distribution of strategies μ) cannot involve strategies that are **dominated** for some player $j \in \mathcal{I}$.

2.11 Concluding remarks

In this chapter we set the theoretical underpinnings for a new framework for the study of games with many players. The cornerstone of our foundations consists of the concepts of distributional form of a player-set, and completeness of a space of games. The distributional form reduces a player-set into a probability distribution with support that specific player-set. This allows us to compare player-sets and games with different number or different labels of players. Completeness of a space of games implies that certain desirable properties are preserved as the number of players increases. These desirable properties involve equilibrium behavior regarding strategic dominance as the number of players increases, and endow our analytical approach with consistency in terms of what is considered to be a plausible outcome of a game with many players.

Our goal is to formulate a basis for a straightforward framework with three basic characteristics: intuition, consistency, and functionality. We believe that we have achieved all three goals.

i. Intuition: the properties of our tools follow from their definitions and we can dispense with elaborate constructs and proofs. We provide a number of examples that illustrate our tools and reinforce understanding of the underlying mechanism.

ii. Consistency: we build our set of tools virtually from scratch[5] and at each step we establish results that test and confirm its structural integrity.

iii. Functionality: at each building step we provide fully worked-through examples of strategic environments amenable to analysis using our tools, and explain exactly how our tools apply to these environments.

The bottom line is that, in addition to the tools that we developed, we also developed an analytical approach based on complete spaces of games with countably-many players, where Nash equilibrium constitutes a plausible prediction of how the modeled strategic situation will be resolved.

2.12 Appendix

2.12.1 Proof of Proposition 4

The metric space $(\Delta(\mathcal{I}), d_{\Delta(\mathcal{I})})$ *of player-sets in distributional form is complete.*

Proof. The result follows as soon as we notice that $\Delta(\mathcal{I})$ is a compact and totally bounded metric space, and as such it is also complete.[6] □

[5] Our set of tools is constructed from scratch as regards the economic theory and game theory; inevitably and unregrettably we rely heavily on mathematical results of topology and measure theory.

[6] In fact, every compact metric space is also totally bounded but for the sake of clarity

2.12.2 Proof of Proposition 5

The metric space $(\mathcal{G},\ d_g)$ *is complete.*

Proof. Note that the Hausdorff metric that defines d_g involves only the product metric d_1. Since the Hausdorff metric inherits completeness (Munkres (2000)), in order to prove that $(\mathcal{G},\ d_g)$ is complete it is sufficient to show that

$$d_1 \text{ is complete on } \mathcal{I} \times \Delta(\mathcal{I}) \times A^2 \times \Delta(A)$$

If a metric space can be expressed as the product of finitely-many complete spaces, then by a corollary of Tychonoff's Theorem (Aliprantis and Border (2005)) also this metric space is complete. Hence, to show that d_1 is a complete metric on $\mathcal{I}\times\Delta(\mathcal{I})\times A^2\times\Delta(A)$ it is sufficient to show that each metric involved in this product metric is complete on its respective space.

i. The metric space $(\mathcal{I},\ d_{\mathcal{I}})$ is compact by construction and since every compact metric space is also totally bounded, this space is also complete.

ii. By Proposition 4 the metric space $(\Delta(\mathcal{I}), d_{\Delta(\mathcal{I})})$ is complete.

iii. The metric space $(A,\ d_A)$ is compact by assumption and therefore by Theorem 6.4 in Parthasarathy (1967) the metric space $(\Delta(A),\ d_{\Delta(A)})$ is also compact and hence both are complete.

Therefore d_1 is complete on $\mathcal{I} \times \Delta(\mathcal{I}) \times A^2 \times \Delta(A)$ and consequently $(\mathcal{G},\ d_g)$ is complete. □

we mention here both properties.

2.12.3 Proof of Proposition 6

The metric space $(\mathcal{B}, d_2)$ *of graphs of best-response correspondences is a complete metric space with respect to the product metric*

$$d_2 := \frac{d_{\mathcal{I}} + d_{\Delta(\mathcal{I})} + d_A + d_{\Delta(A)}}{4} \quad \text{on} \quad \mathcal{B} := \mathcal{I} \times \Delta(\mathcal{I}) \times A \times \Delta(A)$$

Proof. In Proposition 5 we proved that $(\mathcal{I},\ d_{\mathcal{I}})$, $(\Delta(\mathcal{I}),\ d_{\Delta(\mathcal{I})})$, $(A,\ d_A)$, and $(\Delta(A),\ d_{\Delta(A)})$ are complete metric spaces. Then by a corollary of Tychonoff's Theorem (Aliprantis and Border (2005)) also the metric space $(\mathcal{B},\ d_2,)$ is complete.

$\square$

Chapter 3

Complete Spaces of Games with Types

3.1 Introduction

In this chapter we apply the tools that we developed in Chapter 2 to games where players can possibly be grouped into "types". Each player has a type that summarizes attributes that are relevant to the game. Harsanyi (1967) introduced the concept of player-types in game theory by assuming that the players are randomly drawn from certain populations of individuals of possibly different characteristics or attribute vectors. Nevertheless, Harsanyi (1967) proposed player-types in an *incomplete* information setting while we are using player-types in a *complete* information setting. A simple example would be the case of a government that wishes to impose a tax but cannot fully customize the tax amount to each individual's unique characteristics due to the associated complexity cost. Therefore, the government implements a tax-plan based on three "types" of individuals: "married with children", "married without children", and "single".

In this chapter we map the player-space $\mathcal{I}$ to a type-space $\mathcal{T}$ which is a non-empty countably-infinite closed and metrizable set, exactly as $\mathcal{I}$. We can readily extend many of our definitions and constructs, and we do not need to go through every single step as we did in Chapter 2. Even though we move faster in sections that share the same motivation with those of Chapter 2, we do provide fully worked-through examples of how our concepts and tools apply to strategic environments with possibly different types of players.

We underline that our study involves exclusively *non-cooperative* games where players are not allowed to cooperate with each other in their choice of strategy. Every player contemplates unilateral moves and deviations of herself and not of a group of players. Because of non-cooperation the bearing of some of our tools on the case of games with types is limited. Below we explain how and why.

In Chapter 2 we defined the distributional form of a player-set. This concept applies in a qualitatively new way to the case of players with types. The distributional form of a player-set with types captures the *proportion* of players that are of each type. Two or more players can be of the same type and our examples demonstrate also the case of non-equiprobable measures.

Nonetheless, in a game in distributional form, typically players' equilibrium behavior will be independent of the possibility of players being grouped into types. This happens because players are simply not allowed to cooperate with players of the same type. At equilibrium, players of the same type may choose the same strategy exactly because they are of the same type but this coincidence is *not* part of their assumed strategic behavior. Their decision process does not include the fact they belong to a group of players with the same type. On that account, our analysis of Chapter 2 applies directly to the case of games with types.

In Section 2 we define the space of players' types and explain how our analytical approach applies to games with countably many types of players. Given a set of players we define a "type-profile" which is a list of the players' types, and a "type-profile in distributional form" which consists of a probability measure on the space of types. This probability measure singles-out the types that are present in the type-profile by assigning non-zero weight to the types present in the type-profile, and zero weight to other types.

In Section 3 we provide an example of a complete metric space of firms with types, and in Section 4 an example of an unrestricted and a restricted game with types using type-profiles in distributional form.

3.2 The players' space and types' space

In Chapter 2 we identified players according to their name or "label". The space of all potential players was denoted by

$$\mathcal{I} := \{1, \frac{1}{2}, ..., \frac{m}{n}, ... \mid m < n \text{ and } m, n \in \mathbb{N}\} \cup \{0\}$$

and its elements where the labels of the players. In this chapter we enrich the space of players by allowing players to be grouped into possibly different "types". We assume that each player $i \in \mathcal{I}$ can be sufficiently described by her type $t^i \in \mathcal{T}$ which contains all relevant information including the player's preferences over outcomes and the way that her actions affect the payoff of other players. Players $i, j \in \mathcal{I}$ are considered to be of the same type if $t^i = t^j$

for $t^i, t^j \in \mathcal{T}$.

To this purpose we define a "type-space" of players

$$\mathcal{T} := \{t^1, t^{\frac{1}{2}}, ..., t^{\frac{m}{n}}, ... \mid m < n \text{ and } m, n \in \mathbb{N}\} \cup \{0\}$$

which is a non-empty countably-infinite closed and metrizable set and contains the types of players in $\mathcal{I}$. In our case $\mathcal{T}$ is a compact metrizable space by the Heine-Borel Theorem (Aliprantis and Border (2005)) since it is closed, bounded, and metrizable.

In the following definition we make clear the relationship between the player-sets of Chapter 2 and the type-profiles of this chapter.

Definition 29. *Given a set of players*

$$N := (1, \frac{1}{2}, .., \frac{m}{n}, ...) \subseteq \mathcal{I}$$

*its **type-profile** is the set*

$$T_N := (t^1, t^{\frac{1}{2}}, ..., t^{\frac{m}{n}}, ...) \subseteq \mathcal{T}$$

where $t^i \in T_N$ is the type of player $i \in N$.

For each type-profile $T_N \subset \mathcal{T}$ we define a Borel probability measure $\lambda_{T_N} \in \Delta(\mathcal{T})$ that satisfies the following:

1. the support of λ_{T_N} consists of at most $\#T_N$ atoms (the types of players),
2. $\lambda_{T_N}(\{t^i\}) \in (0, 1]$ for each $t^i \in T_N$,
3. $\lambda_{T_N}(\{t^i\} \notin T_N) = 0$ and $\lambda_{T_N}(T_N) = 1$.

Since $\mathcal{T}$ is a compact metrizable space the support of $\lambda_N \in \Delta(\mathcal{T})$ always exists (Chapter II Parthasarathy (1967) and Theorem 12.14 Aliprantis and Border (2005)). Endowed with the weak-convergence topology $\Delta(\mathcal{T})$ is a compact space like $\mathcal{T}$ (Theorem 15.11 Aliprantis and Border (2005)). The associated Prohorov metric is denoted by $d_{\Delta(\mathcal{T})}$.

In the following definition we make precise the notion of a type-profile in distributional form. Two different type-profiles may involve not only different *types* of players but also different *number* of types. The distributional form maps each type-profile to a pair consisting of the cardinality of the set and a probability measure on $\mathcal{T}$. This way, the distributional form provides a convenient way of comparing type-profiles of different size and contents.

Definition 30. *Given a type-space $\mathcal{T}$, the* ***distributional form*** *of a type-profile $T_N \subseteq \mathcal{T}$ consists of a probability measure $\lambda_{T_N} \in \Delta(\mathcal{T})$ whose support is the type-profile T_N. The* ***weight*** *of type $t^i \in \mathcal{T}$ is a positive number less than* 1 *if $t^i \in T_N$ and* 0 *if $t^i \notin T_N$.*

From here on we consider exclusively type-profiles such as

$$T_N := (t^1, t^{\frac{1}{2}}, ..., t^{\frac{m}{n}}, ...)$$

and we can in effect ignore the corresponding player-sets

$$N := (1, \frac{1}{2}, .., \frac{m}{n}, ...)$$

without losing any information.

We simplify notation by

(a) denoting the type profile T_N simply by N, the related player-set that generated T_N

(b) using simply j instead of t^j to denote the type of player j

Note that according to our definition, players $i, j \in \mathcal{I}$ are considered to be of the same type if $t^i = t^j$ for $t^i, t^j \in \mathcal{T}$. Nonetheless, with our simplification of notation in (b) we should allow for $i, j \in \mathcal{T}$ to possibly be of the *same* type even though numerically it may be the case that $i \neq j$. For example, 1 and $\frac{1}{2}$ may represent the same type even though $1 \neq \frac{1}{2}$. This could happen in theory but, to minimize confusion, in our examples we use the same number to represent two players of the same type and avoid using different numbers to represent the same type.

3.2.1 Example: Sets of firms with types

Consider $\mathcal{T} := \{1, ..., \frac{m}{n}, ... \mid m < n \text{ and } m, n \in \mathbb{N}\} \cup \{0\}$ denoting the types of firms in a market game. First we consider a type-profile B where each type appears at least twice. The restricted type-profile of firms

$$B := \{1, 1, \frac{1}{2}, \frac{1}{2}, ..., \frac{9}{10}, \frac{9}{10}\} \subset \mathcal{T}$$

has type-profile distributional form λ_B with

$$\lambda_B(\{1\}) = \lambda_B(\{\frac{1}{6}\}) = \lambda_B(\{\frac{m}{n} \mid 0 < m < n \leq 10 \text{ and } m, n \in \mathbb{N}\}) = \frac{2}{92}$$

since each of these types appears twice in B, and

$$\lambda_B(\{\frac{1}{4}\}) = \lambda_B(\{\frac{1}{5}\}) = \lambda_B(\{\frac{2}{5}\}) = \lambda_B(\{\frac{3}{5}\}) = \lambda_B(\{\frac{4}{5}\}) = \frac{4}{92}$$

since each of these types appears four times in B, and

$$\lambda_B(\{\frac{1}{2}\}) = \frac{10}{92} \text{ since } \frac{1}{2} \text{ appears ten times in } B$$

and

$$\lambda_B(\{\frac{1}{3}\}) = \frac{6}{92} \text{ since } \frac{1}{3} \text{ appears six times in } B$$

and 0 weight is assigned to types not present in B.

Consider a different restricted type-profile of firms

$$K := \{1, 1, 1, 1, 1, \frac{1}{4}, \frac{1}{9}, ..., \frac{m^m}{n^n}, ...\frac{9^9}{10^{10}}\} \subset \mathcal{T}$$

The distributional form of K is $\lambda_K)$ with

$$\lambda_K(\{\frac{m^m}{n^n} \mid m < n < 10 \text{ and } m, n \in \mathbb{N}\}) = \frac{1}{50}$$

and

$$\lambda_K(\{1\}) = \lambda_K(\{\frac{1}{4}\}) = \frac{5}{50} \text{ since type 1 appears five times}$$

and

$$\lambda_K(\{\frac{1}{9}\}) = \lambda_K(\{\frac{4}{9}\}) = \frac{3}{50} \text{ since each of these types appears three times}$$

and

$$\lambda_K(\{\frac{1}{16}\}) = \lambda_K(\{\frac{9}{16}\}) = \lambda_K(\{\frac{1}{25}\}) = \frac{2}{50} \text{ since each of these types appears twice}$$

and 0 weight is assigned to types not present in K. Note that $B \not\subset K$ since the type $\frac{1}{2}$ belongs to B but not to K also $K \not\subset B$ since the type $\frac{9^9}{10^{10}}$ belongs to K but not to B.

Now we can proceed to compare the two type-profiles and see the advantages of the distributional form. The original type-profiles B and K not only have not equal cardinality but also have different contents (different players). On the other hand, λ_B and λ_K are both probability measures and comparison is possible.

3.3 Cournot market game with types in distributional form

Here we apply our definition of a game in distributional form (Section 2.5) to the case of a Cournot market game with possibly different types of firms. Consider the limiting Cournot market game with firm-types

$$G_{\mathcal{T}} = \langle \lambda, A, p \rangle$$

Each firm with type $j \in \mathcal{T}$ chooses a level of production of a single homogeneous good

$$q^j \in A^j \subseteq A := [0, q_{max}] \cap \mathbb{Q}$$

and faces $c \in [0, 1)$ unit cost of production. Assume that $P(Q) = 1 - Q$ relates the price P of the good with the average quantity available $Q := \int_{\mathcal{T}} q^i d\lambda$. The payoff function of a firm $j \in \mathcal{T}$ can be identified with its profit function

$$p^j(q^j, \int_{\mathcal{T}} q^i d\lambda; c) := (1 - \int_{\mathcal{T}} q^i d\lambda - c)q^j$$

A restricted Cournot market game with firm-types is

$$G_K = \langle \lambda_K, A, p_K \rangle$$

with λ_K as described previously.

The action space is the same as in the previous example, and $P(Q) = 1 - Q$ relates the price P of the good with the average quantity available $Q := \frac{1}{50}\sum_{i \in K} q^i$. Then payoff function of firm with type j is now

$$p^j(q^j, \frac{1}{50}\sum_{i \in K} q^i; c) := (1 - \frac{1}{50}\sum_{i \in K} q^i - c)q^j$$

We see that in both the unrestricted $G_{\mathcal{T}}$ and in the restricted game G_K all parameters of a game: the space of participants, the action-space, and the payoff function of each participant, are defined and consequently both games are well-defined.

3.4 Concluding Remarks

In this chapter we demonstrated how our analytical approach of Chapter 2 applies to games where players can possibly be grouped into different types. The type of a player summarizes her attributes that are relevant to the game. The results of Chapter 2 apply directly to the analysis of games with types: instead of having participating *players* in the game, in Chapter 3 we have participating *types* of players. This is mainly because we work in a space of participants that has the same mathematical properties as that of Chapter 2: a non-empty countably-infinite closed and metrizable space.

We defined all relevant type-profiles, distributional forms, and presented fully worked-through examples of applications of our analytical tools. We skipped the parts where the analysis of Chapter 2 applies directly and focused on the parts that are qualitatively different from Chapter 2.

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